CHAPTER 17

Interest Rate Products: Futures and Options

Interest rate derivative contracts seem less in the spotlight than are derivatives on stocks and stock indexes. One reason is that the markets for bonds are less active than the market for stocks. Do not be misled, however. The bond markets in the United States are, in fact, larger than stock markets. Of the $34.34 trillion in market value of stocks and bonds outstanding in the United States at the end of 2003, about 56% was bonds. It should not be surprising, therefore, that interest rate risk management is a primary concern for corporations, agencies, municipalities, and governments. Indeed, more than two-thirds of all OTC derivatives traded worldwide are written on interest rate instruments.

The first interest rate derivative contract on an exchange appeared 30 years ago, when the CBT introduced futures contracts on GNMA pass-through certificates. Futures contracts on U.S. Treasury bonds, notes, and bills quickly followed. Options on interest rate instruments were launched in late 1982. Even though many of these markets have become incredibly active by exchange standards, the greatest success story is the OTC interest rate swap market. The first interest rate swap was consummated in 1981. Today, about 20 years later, interest rate swaps account for more than half the notional amount of *all* derivatives outstanding worldwide. The interest rate products discussion is divided into two chapters. This chapter focuses on futures and options contracts, that is, contracts with a single future cash flow. The next chapter focuses on contracts with multiple future cash flows. In it, we discuss interest rate swaps, caps, collars, floors, and swaptions.

Our discussion of exchange-traded products in this chapter has three sections. In the first, details of selected exchange-traded contracts and contract markets are provided. Section two provides the principles of interest rate derivatives valuation. For the most part, the principles and valuation methods of Chapters 4 through 9 can be applied directly, with two notable exceptions. First, the no-arbitrage price relation for the CBT's T-bond futures must be modified to account for the fact that the seller has an option to deliver any one of a number of eligible bond issues. Second, for options on short-term debt instruments, the log-normal price distribution assumption is inappropriate. The price of a T-bill, for example, can never exceed its par value. Consequently, we are required to develop a new methodology for valuing interest rate options. To do so, we invoke the assumption that the short-term interest rate is log-normally distributed, and then modify the valuation methods of Chapters 7 through 9. Section three illustrates three important interest rate derviatives risk management strategies—a short-term long hedge, a long-term short hedge, and asset allocation.

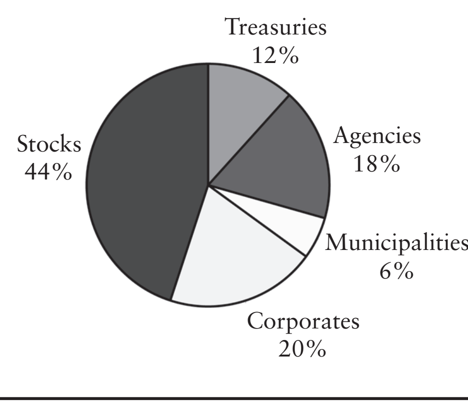
MARKETS

To place the development of interest rate derivatives markets in context, it is useful to get a sense for the underlying asset market. Unlike stocks, bonds are not actively traded on exchanges. They trade in over-the-counter markets, which do not have the transparency of stock markets. As a consequence, the public often perceives the bond market to be smaller and less important than the stock market. Nothing is further from the truth, however. Figure 17.1 shows the market value of stocks and bonds traded in the United States as of December 31, 2003. Of the $34.34 trillion in outstanding securities, 56% are bonds and 44% are stocks. Corporate bonds are the single largest bond market, accounting for 20% of security value. Agencies are the second largest group at 18%, Treasuries account for 12%, and municipalities account for 6%. Below the evolution of interest rate derivatives is discussed. While exchange-traded interest rate derivatives markets are active, the trading volume now pales by comparison to the OTC market.

Evolution of Interest Rate Derivatives

Derivatives contracts on interest rate instruments began trading in the mid-1970s. The first interest rate futures contract, introduced in the fall of 1975, was the Chicago Board of Trade's (CBTs) futures on GNMA Collaterialized

FIGURE 17.1 Market values of bonds and stocks outstanding in the United States as of December 31, 2003. Total market value of all securities is $34.34 trillion.



Source: Information compiled from *Flow of Fund Accounts of the United States* (fourth quarter 2003), www.federalreserve.gov.

Despositary Receipts (CDRs). The CDRs are pools of mortgages whose payments are insured by the Government National Mortgage Association (GNMA), a U.S. governmental agency. They are sometimes referred to as "pass-through" certificates because the payments of the mortgage holders passthrough to the holders of the certificates in the form of coupon interest payments. Prepayments by mortgagors, and prepayments by insurers in the case of default, also pass-through. The coupon rates on the certificates are 0.5% below the rate on the mortgages to cover the 0.44% retained by the servicer who collects and distributes the mortgage payments and the 0.06% paid to GNMA for insuring the pool against default. Different pools of mortgages, even ones with the same coupon rate, behave quite differently from one another due to different rates of prepayment. The futures contract permitted different coupon rates to be delivered, and had an imperfect system for translating these eligible bonds into an 8% coupon bond issue. Without a well-defined underlying asset, arbitrage between the futures and cash market is impeded, and the correlation between the futures and mortgage-backed securities is low, undermining the contract's effectiveness as a hedging vehicle. The contract was delisted in the late 1980s.[1]

The next interest rate futures contracts to be introduced were the Chicago Mercantile Exchange's (CMEs) T-bill futures contract in January 1976 and the CBT's U.S. Treasury bond futures in August 1977. Spurred by success of these contract markets, interest rate futures began to appear on other exchanges worldwide. The Sydney Futures Exchange, for example, introduced futures on 90-day Bank Accepted bills in October 1979, and the London International Financial Futures Exchange introduced trading on long gilt[2] futures in November 1982. Back in the United States, another important innovation occurred in December 1981 when the CME introduced the Eurodollar futures contract. This marked the first time an interest rate futures specified cash-settlement rather than physical delivery.

Options on interest rate instruments first appeared in late 1982. On October 1, 1982, the CBT and the CME simultaneously launched trading of option contracts on T-bond futures and Eurodollar futures, respectively. The Chicago Board Options Exchange (CBOE) introduced options on Treasury bonds and the American Stock Exchange (AMEX) introduced options on Treasury notes and bills on October 22, 1982. Interestingly, options written on interest rates futures are far more actively traded on exchanges than are options on debt instruments directly. One possible reason for this phenomenon is that there are just too many debt issues (for the U.S. Treasury alone) for each to have an actively traded market on an exchange. OTC option dealers, on the other hand, stand ready and willing to create an option on any bond issue that a customer wants.

The last noteworthy event in terms of the evolution of interest rate derivatives markets is the development of the swap market in the early 1980s. An *interest rate swap* is an agreement between two parties to exchange or "swap" a series of periodic interest payments. The most common interest rate swap is to exchange payments on fixed rate debt for floating rate debt. Such a swap is called

[1] Johnston and McConnell (1989) provide an interesting retrospective on the rise and fall of the CBT's GNMA contract market.

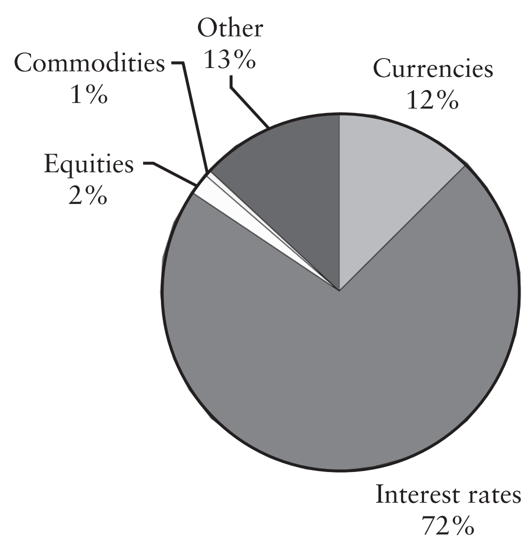
[2] A *gilt* (or gilt-edged stock) is a bond issued and guaranteed by the British government.

a *plain-vanilla interest rate swap*. An early example occurred in 1982, when Sallie Mae swapped the interest payments on intermediate-term fixed rate debt for floating rate payments indexed to the three-month T-bill yield. In the same year, a USD 300 million seven-year Deutsche Bank bond issue was swapped into USD LIBOR. While today, OTC swaps are written on a number of different types of underlying assets, interest rate swaps are far and away the largest asset category. As of yearend 2003, interest rate derivatives accounted for 72% of the notional amount of all OTC derivatives outstanding. (See Figure 17.2.) Of this amount, more than 76% of interest rate derivatives were swaps, with the remaining 24% being between split options (14.7%) and forwards (10.3%). (See Figure 17.3.) We return to OTC interst rate products in the next chapter.

Interest Rate Futures

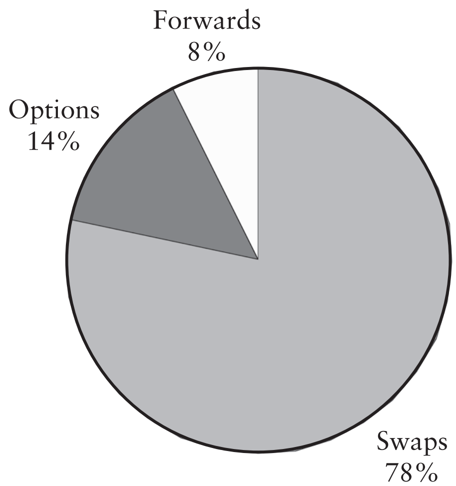
In the case of stock, stock index, and foreign currency products discussed in the last three chapters, there is a single source of risk underlying the derivatives contracts (i.e., the price risks of a stock, stock index, or foreign currency). Interest rate derivatives, on the other hand, are more intriguing in the sense that interest rate risk is often categorized by whether it is short-term, intermediate-term, or long-term, with the behavior of each interest rate being quite different. By far the most active short-term interest rate contract is the CME's three-month Eurodollar futures. The CBT's T-bond futures captures most of the trading in the long-term arena, and the CBT's 10-year and five-year contracts capture the intermediate term. All other interest rate futures contract trading volumes of the other contracts pale by comparison.

FIGURE 17.2 Percentage of total notional amount of derivatives outstanding worldwide on December 2003 by underlying asset category. Total notional amount of derivatives is USD 197.2 trillion.



Source: Information compiled from Bank for International Settlements (www.bis.org), *BIS Quarterly Review*, June 2004.

FIGURE 17.3 Percentage of total notional amount of single-currency, interest rate derivatives outstanding worldwide on December 2003 by contract type. Total notional amount of interest rate derivatives is USD 141.99 trillion.



Source: Information compiled from Bank for International Settlements (www.bis.org), *BIS Quarterly Review*, June 2004.

Eurodollar Futures The contract specifications for the CME's Eurodollar futures are given in Table 17.1. The underlying instrument is a USD 1,000,000 Eurodollar deposit with three months to maturity. Its price is quoted as an index level and is created by subtracting the Eurodollar rate from 100. A price of 94.50, therefore, means that the contract buyer is willing to lend USD 1,000,000 at 5.50% for a three-month period beginning on the date the futures contract expires. Unlike many other interest rate futures, the Eurodollar futures is cash settled at expiration, which is set as the second London business day immediately preceding the third Wednesday of the contract month. The cash settlement price is the British Bankers Association (BBA) Interest Settlement Rate. At 11 AM London time, 16 BBA designated banks provide quotes that reflect their perception of the rate at which U.S. dollar deposits are generally available in the marketplace. The highest four and the lowest four are eliminated. The average of the remaining eight quotes (rounded to the fifth decimal place) is the fixing of the day.

Table 17.2 contains a summary of Eurodollar futures trading on Thursday, March 30, 2006. The June 2006 futures has a reported settlement price of 94.975. This means the promised rate on a three-month Eurodollar time deposit beginning on June 19, 2006 (second London business day immediately preceding the third Wednesday of the contract month) is 5.025%. As noted in Chapter 2, this nominal Eurodollar interest rate is a simple interest rate based on a 360-day year. To find the continuously compounded rate of return, we must undo the nominal rates and the 360-day banker's year converntion. Since

$$e^{r(92/365)} = 1 + 0.05025\left(\frac{92}{360}\right)$$

TABLE 17.1 Selected terms of CME's Eurodollar futures contract.

Exchange	Chicago Mercantile Exchange
Contract unit	$1,000,000 Eurodollar time deposit
Price quote	An index created by subtracting the Eurodollar rate from 100. (e.g., 94.50 implies a 5.50% Eurodollar interest rate.)
Tick size	0.005 (1/2 basis point)
Tick value	$12.50 per contract
Trading hours	7:20 AM to 2:00 PM (CT) Monday through Friday. GLOBEX: Monday through Thursday, 5 PM to 4 PM; Sundays and holidays, 5 PM to 4 PM.
Contract months	40 months on March quarterly cycle (Mar., Jun., Sep., Dec.) plus four nearest serial months.
Last day of trading	Second London bank business day immediately preceding the third Wednesday of the contract month. If it is a bank holiday in New York City or Chicago, trading terminates on the first London bank business day preceding the third Wednesday of the contract month. If an Exchange holiday, trading terminates on the next preceding business day common to London banks and the Exchange.
Settlement	Cash settlement price determined by the British Bankers Association (BBA) Interest Settlement Rate. At 11 AM London time, 16 BBA designated banks provide quotes which reflect their perception of the rate at which U.S. dollar deposits are generally available in the market place. The four highest and the four lowest are eliminated. The average of the remaining eight quotes (rounded out to the fifth decimal place) is the fixing for the day.

the continuous rate is

$$r = \frac{\ln(1 + 0.05025(92/360))}{92/365} = 5.2425\%$$

The function, OV_IR_CONV_ED_YIELD(*rate*,*days*), performs this transformation, where *rate* is the nominal Eurodollar interest rate and *days* is the number of days to maturity. In the present context,

OV_IR_CONV_ED_YIELD(0.05025,92) = 0.052425

Table 17.2 also shows a phenomenon that is uncommon to most exchange-traded derivatives, that is, contract months extend out into the future 10 years. For most exchange-traded derivatives with the March quarterly expiration (March, June, September, December), the nearby contract has the greatest trading volume, with the second nearby contract having only a small fraction of that of the nearby. For the Eurodollar contracts, even the June 2011 futures has 2,002 contracts outstanding. At a contract denomination of USD 1,000,000, this represents over USD 2 billion. The reason for the activity ranging out so far is that this is the primary market that OTC market makers use to hedge their floating rate

TABLE 17.2 Summary of trading activity of Chicago Mercantile Exchange's Eurodollar futures on March 30, 2006.

Month	Open	High	Low	Last	Settlement	Point Change	Estimated Volume	Open Interest
Apr-06	94.96	94.96	94.945	94.9475B	94.95	–1.25	5,693	44,588
May-06	94.87	94.96B	94.86A	94.865A	94.86	–2	65	10,112
Jun-06	94.825	94.84	94.79	94.795	94.795	–3	327K	1365K
Jul-06	94.785	94.79	94.765A	94.77A	94.765	–3	748	1,043
Aug-06	—	—	94.745A	94.76A	94.745	–3	493	
Sep-06	94.77	94.79	94.715	94.72	94.725	–4	467K	1340K
Dec-06	94.785	94.81	94.725	94.74	94.74	–4	431K	1388K
Mar-07	94.845	94.865	94.775	94.785	94.79	–5	382K	1098K
Jun-07	94.885	94.91	94.815	94.83	94.83	–5	322K	898,819
Sep-07	94.905	94.93	94.835	94.85	94.855	–4.5	188K	786,126
Dec-07	94.9	94.925	94.84	94.85	94.855	–4.5	124K	549,025
Mar-08	94.905	94.92	94.83	94.845	94.85	–4.5	78K	356,216
Jun-08	94.88	94.9	94.81	94.835B	94.83	–4.5	27K	253,736
Sep-08	94.86	94.88	94.79	94.815B	94.81	–4.5	21K	213,770
Dec-08	94.81	94.84	94.755A	94.775B	94.77	–4.5	19K	179,193
Mar-09	94.82	94.82	94.74	94.76	94.755	–4.5	13K	129,372
Jun-09	94.79	94.79	94.70A	94.73	94.73	–4.5	10K	116,373
Sep-09	94.765	94.765	94.67A	94.7	94.7	–5	7,654	107,602
Dec-09	94.685	94.72B	94.63A	94.66	94.66	–5	8,744	95,764
Mar-10	94.675	94.71B	94.615A	94.645	94.645	–5.5	6,926	78,810
Jun-10	94.63	94.69B	94.59A	94.625A	94.625	–5.5	5,255	60,504
Sep-10	94.615	94.665B	94.565A	94.6	94.595	–6	3,842	54,679
Dec-10	94.635	94.635	94.52	94.565	94.56	–6	2,859	54,141
Mar-11	94.625	94.625	94.52A	94.555	94.55	–6	3,712	23,939
Jun-11	—	—	94.50A	94.525B	94.53	–6	2,002	11,926
Sep-11	—	—	94.48A	94.505B	94.505	–6.5	202	16,176
Dec-11	—	—	94.45A	94.475B	94.475	–6.5	202	12,631
Mar-12	—	—	94.445A	94.47B	94.47	–6.5	202	9,374
Jun-12	94.49	94.49	94.42A	94.44B	94.445	–6.5	12	5,904
Sep-12	94.47	94.47	94.40A	94.42B	94.42	–7	11	7,659
Dec-12	94.44	94.44	94.37A	94.39B	94.39	–7	5	5,927
Mar-13	—	—	94.365A	94.385B	94.385	–7	2	2,883
Jun-13	—	—	94.34A	94.355B	94.36	–7	2	2,069
Sep-13	—	—	94.315A	94.335B	94.335	–7	2	1,336
Dec-13	—	—	94.285A	94.30B	94.305	–7	2	1,220
Mar-14	—	—	94.28A	94.295B	94.3	–7	2	1,400
Jun-14	—	—	94.255A	94.27B	94.275	–7	2	466
Sep-14	—	—	94.23A	94.245B	94.25	–7	2	641
Dec-14	—	—	94.20A	94.215B	94.22	–7	2	1,198
Mar-15	—	—	94.195A	94.21B	94.215	–7	2	392
Jun-15	—	—	94.175A	94.19B	94.19	–7.5	2	488
Sep-15	—	—	94.155A	94.17A	94.165	–8	2	565
Dec-15	—	—	94.125A	94.14A	94.135	–8	2	260
Mar-16	—	—	94.12A	94.135A	94.13	–8	2	230

Source: Data drawn from www.cme.com.

risk exposure on interest rate swaps. We discuss the linkage between these markets in the interest rate swap valuation section later in the next chapter.

U.S. T-Bond and T-Note Futures The contract specifications of the CBT's T-bond and 10-year T-note contracts are shown in Tables 17.3 and 17.4, respectively. Both requires the delivery of a $100,000 U.S. Treasury coupon-bearing instrument, however, in the case of the T-bond futures it is a T-bond with at least 15 years to maturity or, if the bond is callable, to first call date,[3] and in the case of the 10-year T-note futures it is a T-note with at least 6½ years but less than 10 years to maturity. Both contracts follow a quarterly expiration cycle. Delivery may take place at any time during the delivery month at the discretion of the short. The last day of trading of the futures contract is the eighth last business day of the contract month. Table 17.5 shows the prices of the T-bond and T-notes futures as of the close of trading on March 30, 2006. Note that like their underlying instruments, T-bond and T-note futures have their prices quoted in 32nds. The June 2006 T-bond futures settlement price of 109-06 means that the price is 109.1875% of par.

TABLE 17.3 Selected terms of CBT's U.S. Treasury bond futures contract.

Exchange	Chicago Board of Trade
Contract unit	One U.S. T-bond with a face value of $100,000
Deliverable grades	Any U.S. T-bond with at least 15 years to maturity or to first call date from the first day of the delivery month. Invoice price equals settlement price times a conversion factor plus accrued interest. The conversion factor is the price of the delivered bond to yield 6%.
Price quote	Points ($1,000) and 1/32 of a point. E.g., 80-16 equals 80 16/32.
Tick size	1/32 of a point
Tick value	$31.25 per contract
Trading hours	Open outcry: 7:20 AM to 2:00 PM CT, Monday through Friday Electronic: 7:00 PM to 4:00 PM CT, Sunday through Friday
Contract months	Four contract months on March quarterly expiration cycle (Mar., Jun., Sep., Dec.)
Last day of trading	Seventh last day preceding the last business day of the delivery month.
Settlement	Physical delivery
First delivery day	First day of the contract month
Last delivery day	Last day of contract month

[3] The U.S. Treasury stopped issuing 30-year bonds in November 2001, an era when the government was running budget surpluses—not deficits—and financing the government's debt was easier. Consequently, the deliverable supply of long-term U.S. Treasury bonds had been declining. In August 2005, the U.S. Treasury announced that it would once again issue 30-year bonds beginning February 2006.

For decades, the 30-year bond served as a closely followed benchmark for the entire fixed-income market. But Treasury stopped issuing the long bond in 2001, in an era when the government was finally running budget surpluses—not deficits—and financing the government's debt was easier.

TABLE 17.4 Selected terms of CBT's 10-year U.S. Treasury note futures contract.

Exchange	Chicago Board of Trade
Contract unit	One U.S. T-note with a face value of $100,000
Deliverable grades	Any U.S. T-note with at least 6-1/2 years but not more than 10 years to maturity from the first day of the delivery month. Invoice price equals settlement price times a conversion factor plus accrued interest. The conversion factor is the price of the delivered bond to yield 6%.
Price quote	Points ($1,000) and one-half of 1/32 of a point. E.g., 80-165 equals 80 16.5/32.
Tick size	One-half of 1/32 of a point
Tick value	$15.625 per contract
Trading hours	Open outcry: 7:20 AM to 2:00 PM CT, Monday through Friday Electronic: 7:00 PM to 4:00 PM CT, Sunday through Friday
Contract months	Four contract months on March Quarterly expiration cycle (Mar., Jun., Sep., Dec.)
Last day of trading	Seventh last day preceding the last business day of the delivery month.
Settlement	Physical delivery
First delivery day	First day of the contract month
Last delivery day	Last day of contract month

TABLE 17.5 Summary of trading activity of Chicago Board of Trade's Treasury bond and 10-year Treasury note futures on Thursday, March 30, 2006.

Contract month	Open	High	Low	Settle	Chg	Open Interest
Treasury Bonds (CBT)						
Jun-06	109-19	109-21	108-26	109-06	–16	642,750
Sep-06	109-16	109-16	108-29	109-07	–16	3,429
Dec-06				109-19	–16	588
Mar-07				109-14	–16	1
10-Year Treasury Notes (CBT)						
Jun-06	106-200	106-200	106-040	106-100	–0-095	2,045,545
Sep-06	106-210	106-210	106-065	106-110	–0-105	666,465
Dec-06				106-100	–0-115	148
Mar-07				106-100	–0-115	1

Source: Data drawn from www.cbot.com.

Conversion Factor The CBT's Treasury contracts call for the delivery of any Treasury instrument satisfying a particular maturity constraint. As noted, above, the T-bond futures contract calls for the delivery of any $100,000 U.S. Treasury bond with at least 15 years to maturity or to first call date. Different T-bonds, however, have different coupons and therefore different prices. If no other restriction is applied, the individuals who are short the futures would deliver zero-coupon bonds since, holding other factors constant, they have the lowest value.

The CBT's system of conversion is designed to make the short futures indifferent about which one of the eligible bonds to deliver. It does so by converting every bond to a common hypothetical 6% coupon-bearing bond.[4] To illustrate the principle underlying the CBT's system of conversion, consider the price of a 6%, semiannual coupon-bearing bond with 15 years to maturity. If the yield to maturity is 6%, the bond's price may be written

$$B = \sum_{t=1}^{30} \frac{3}{1.03^t} + \frac{100}{1.03^{30}} = 100.00$$

where the coupon rate and yield have been halved to account for the semiannual coupon payment convention. Now consider the price of a 9%, 15-year bond at a 6% yield to maturity, that is

$$B = \sum_{t=1}^{30} \frac{4.5}{1.03^t} + \frac{100}{1.03^{30}} = 129.40$$

Since the only difference between these bonds is their coupon payment, it must be the case that owning the 9% coupon-bearing bond is like owning 1.2940 6% bonds. Since the futures price is based on a 6% coupon bond, the futures price is multiplied by a conversion factor of 1.2940 to compute the amount paid (delivery price) by the long to the short if the short delivers the 9% coupon issue.

The actual formula for computing the CBT's conversion factor is more complicated than what is demonstrated in the above example because coupons are paid on a semiannual basis, and, in general, the next coupon payment is made in less than six months (i.e., we are part of the way through the current coupon period). The actual formula is

$$CF = (1+y/2)^{-X/6}\left\{\frac{C}{2} + \left[\frac{C}{y}(1-(1+y/2)^{-2n}) + (1+y/2)^{-2n}\right]\right\} - \frac{C}{2}\frac{(6-X)}{6} \quad (17.1)$$

where CF is the conversion factor, C is the annual coupon rate of the bond in decimal form, y equals 0.06, n is the number of whole years to first call if the

[4] The reason that the CBT allows T-bonds with different coupons to be delivered is to prevent the possibility of market manipulation. Each T-bond has limited supply. If the futures contract were written on a single T-bond issue, it would be possible for a single individual or firm to corner the market in the underlying bond and attempt a short squeeze. See Chapter 1.

bond is callable or the number of years to maturity if the bond is not callable, and X is the number of months that the maturity exceeds n, rounded down to the nearest quarter (e.g., $X = 0,3,6,9$). Note that if $X = 0,3,6$, the formula (17.1) is used directly. If $X = 9$, set $2n = 2n + 1$, $X = 3$, and calculate as above. The CBT and others publish and distribute conversion factor tables like those shown in Table 17.6. Note that in Table 17.6 the conversion factor of a 9% bond maturing in exactly 15 years is 1.2940, just as we computed earlier.

To illustrate the use of the conversion factor system, assume that we are considering delivering the 9½s of November 2021 on the June 2006 T-bond futures contract. This bond is eligible for delivery because, as of June 1, 2006 (i.e., the first possible delivery date), it has more than 15 years to maturity. In point of fact, on June 1, 2006, the 9½s of November 2021 have 15 years and five months to maturity (i.e., $n = 15$ and $X = 5$). When rounded down to the nearest quarter, we have $X = 3$. Thus, the CBT deems this bond to have 15 years to maturity for delivery purposes. Using Table 17.6, we see that the conversion factor of this bond is 1.3464. In other words, in place of delivering the hypothetical 6% bond underlying the June 2006 futures, we can deliver the 9½s of November 2021, and the buyer is going to have to pay 1.3464 times the prevailing futures price. The conversion factor may also be determined using the OPTVAL function,

OV_IR_CONV_TBOND_CONVFAC(*ncoup*,*nyrs*,*coup*)

where *ncoup* is nominal (annualized) coupon rate (in decimal) specified by the exchange, *nyrs* is the number of years to maturity of the T-bond being delivered, and *coup* is the coupon rate (in decimal) of the T-bond being delivered. Consequently,

OV_IR_CONV_TBOND_CONVFAC (0.06,15.41667,0.0950) = 1.3464

Invoice Price On the delivery date, the seller of the T-bond futures delivers an eligible T-bond to the buyer of the T-bond futures contract. In return, the buyer must pay an amount of money called the *invoice price*. The amount of the invoice price equals the sum of the futures price times the conversion factor of the delivered bond and the accrued interest on the delivered bond. Suppose that on June 1, 2006 (i.e., the first day of the delivery month), for example, the June 2006 futures is priced at 100-17. Like the underlying bonds, the digits to the right of the dash represent 32nds, so the futures price is actually 100.53125. If we sell the futures and then promptly deliver the 9½s of November 2021 to the futures contract buyer, we would receive the invoice price, which equals 100.53125 times 1,000 (the denomination of the futures contract) times 1.3464 (the conversion factor of the 9½s of November 2021 as of June 1, 2006), that is,

$$100.53125 \times \$1{,}000 \times 1.3464 = \$135{,}355.28$$

plus the accrued interest on the 9½s of November 2021 as of June 1, 2006,[5] that is,

[5] As of June 1, 2006, 15 days have elapsed in the current coupon period for the 9½s of November 2021 (from May 15 to June 1), where the current coupon period is 184 days in length (from May 15, 2006 to November 15, 2006).

TABLE 17.6 Sample of CBT conversion factors for T-bonds eligible for delivery on T-bond futures.

Years to Maturity	Coupon Interest Rate								
	9.000%	9.125%	9.250%	9.375%	9.500%	9.625%	9.750%	9.875%	10.000%
15.00	1.2940	1.3063	1.3185	1.3308	1.3430	1.3553	1.3675	1.3798	1.3920
15.25	1.2969	1.3092	1.3216	1.3340	1.3464	1.3587	1.3711	1.3835	1.3959
15.50	1.3000	1.3125	1.3250	1.3375	1.3500	1.3625	1.3750	1.3875	1.4000
15.75	1.3028	1.3154	1.3280	1.3406	1.3533	1.3659	1.3785	1.3911	1.4037
16.00	1.3058	1.3186	1.3313	1.3441	1.3568	1.3695	1.3823	1.3950	1.4078
16.25	1.3085	1.3214	1.3342	1.3471	1.3600	1.3728	1.3857	1.3985	1.4114
16.50	1.3115	1.3245	1.3374	1.3504	1.3634	1.3764	1.3894	1.4023	1.4153
16.75	1.3141	1.3272	1.3403	1.3534	1.3665	1.3795	1.3926	1.4057	1.4188

$$(9.50/2) \times \$1{,}000 \times (15/184) = \$387.23$$

The total invoice price is \$135,355.28 + 387.23 = \$135,742.51.

The conversion factor mechanics are more tedious than they are difficult to understand. As noted earlier, the motivation for making a number of different T-bonds eligible for delivery is to ensure that no single individual or bank can attempt to corner the market in the bond underlying the futures. The system of conversion does not work exactly, however, and one of the eligible bonds winds up being "cheapest-to-deliver" in practice. We will show how to identify this bond and modify the futures pricing relation in the next section of the chapter.

Interest Rate Options

The most active exchange-traded interest rate options are those written on the CME's Eurodollar futures, and the CBT's five-year T-note, 10-year T-note, and T-bond futures. All of these contracts are futures options. Although some exchanges such as the CBOE and the AMEX have tried to develop options on specific bond issues, none of the markets have been particularly successful. That is not say that bond option markets are inactive. They are, but not on exchanges. Recall that Figure 17.3 showed that the notional amount of interest rate options outstanding in the OTC market was about USD 20 trillion at the end of 2003.

Eurodollar Futures Options The contract specifications of the CME's Eurodollar futures options are given in Table 17.7. The options are American-style, and expire together with the underlying futures on the second London business day before the third Wednesday of the contract month. Exercise of a Eurodollar option before expiration results in the delivery of the underlying futures. Thus, if we hold a June 2006 call option written on a Eurodollar futures and exercise it, we will receive a long position in the June 2006 Eurodollar futures at the end of the day and will receive cash proceeds equal to the difference between the futures settlement price and the exercise price of the call.

Eurodollar futures options follow a quarterly expiration cycle like the futures. In addition, there are two serial months. If we are standing at the end of March 2006, for example, the nearby Eurodollar futures will be the June 2006 contract. Eurodollar futures options with April 2006, May 2006, and June 2006 expirations will appear—the April and May contracts being the serial months and the June contract being the quarterly expiration. In this case, the April and May options contracts, like the June contract, are written on the June 2006 Eurodollar futures. Unlike the June contract, which, if carried to expiration is cash-settled, the April and May contracts have delivery settlement.

U.S. T-Bond Futures Options The contract specifications of the CBT's option contracts on the T-bond and T-note futures are similar, so we present only the T-bond futures' contract specifications in Table 17.8. Like the Eurodollar futures options, early exercise results in receiving a position in the underlying futures. Exercising a call, for example, results in a long futures position, and exercising a put results in a short futures position. Unlike the Eurodollar futures options, however, the last

TABLE 17.7 Selected terms of CME's Eurodollar futures options contract.

Exchange	Chicago Mercantile Exchange
Contract unit	$1,000,000 Eurodollar time deposit
Price quote	An index created by subtracting the Eurodollar rate from 100. (e.g., 94.50 implies a 5.50% Eurodollar interest rate.)
Tick size	0.01 (1 basis point)
Tick value	$25.00 per contract
Exercise prices	At 0.25 intervals for the nearest listed expiration in the quarterly cycle, and the serial month expirations with the same underlying futures.
Trading hours	7:20 AM to 2:00 PM (CT), Monday through Friday. GLOBEX: Monday through Thursday 5 PM to 6:50 AM; Sundays and holidays, 5 PM to 6:50 AM.
Contract months	Eight months in March quaterly cycle (Mar., Jun., Sep., Dec.) cycle plus two serial months.
Last day of trading	Quarterly: Options trading shall terminate at 11:00 AM (London Time) 5:00 AM (Chicago Time) on the second London bank business day before the third Wednesday of the contract month. Serial Eurodollar options trading shall terminate on the Friday immediately preceding the third Wednesday of the contract month. If the foregoing date for termination is an Exchange holiday, options trading shall terminate on the immediately preceding business day.
Settlement	Cash settlement price determined by the British Bankers Association (BBA) Interest Settlement Rate. At 11 AM London time, 16 BBA designated banks provide quotes which reflect their perception of the rate at which U.S. dollar deposits are generally available.
Style	American-style
Settlement	The quarterly contracts are cash settled in the same manner as the futures. Early exercise or exercise of serial contracts requires the delivery of the underlying futures position.

day of trading is the first Friday preceding, by *at least* two business days, the first notice day for the corresponding T-bond futures contract. In general, the first notice day of the futures is the first business day of the contract month. Thus although the June 2006 option contract is written on the June 2006 futures, it expires in May 2006, on the first Friday preceding, by *at least* two business days, June 1, 2006. The CBT also lists a "front month" contract, if the front month is not a quarterly expiration. This means that standing at the end of June 2006, there will be a July 2006 T-bond futures option contract, however, it is written on the September 2006 T-bond futures.

NO-ARBITRAGE RELATIONS AND VALUATION

Like common stocks and stock indexes, the no-arbitrage price relations and valuation methods are best modeled using a discrete flow carry cost assumption.

TABLE 17.8 Selected terms of CBT's U.S. Treasury bond futures options contract.

Exchange	Chicago Board of Trade
Ticker symbol	CG for calls/PG for puts
Contract unit	One U.S. T-bond futures (of a specified maturity) having a face value of $100,000
Tick size	1/64 of a point
Tick value	$15.625 per contract
Exercise prices	
	1-point strikes for the nearby contract month in a band consisting of the at-the-money, 4 above, and 4 below; 2-point strikes are listed outside this band. Back months are also listed in 2-point strike price intervals.
Trading hours	Open outcry: 7:20 AM to 2:00 PM CT, Monday through Friday Electronic: 7:02 PM to 4:00 PM CT, Sunday through Friday
Contract months	First three consecutive contract months (two serial expirations and one quarterly expiration) plus the next two months in March quarterly cycle (Mar., Jun., Sep., Dec.). There will always be five months available for trading. Monthlies will exercise into the first nearby quarterly futures contract. Quarterlies will exercise into futures contracts of the same delivery period.
Last day of trading	Options cease trading on the last Friday, preceding by at least two business days, the last business day of the month preceding the contract month.
Style	American-style. Options that expire in-the-money are automatically exercised.
Settlement	Physical delivery of futures position
Trading hours	Open outcry: 7:20 AM to 2:00 PM CST, Monday through Friday

The U.S. Treasury bonds underlying long-term bond futures and options, for example, pay coupons on a semiannual basis, with the amount of the coupon interest payment date as well as the payment date known. The debt instruments underlying short-term bond futures and options (e.g., T-bill and Eurodollar futures and options) make no intermediate coupon payments, in which case the only carry cost is the interest rate.

This section provides a summary of the no-arbitrage price relations and valuation equations of exchange-traded interest rate derivatives. Most of the results are straightforward extensions of the materials developed in Chapters 4 through 9. There are two notable exceptions, however. First, under the no-arbitrage price relations discussion, we are forced to consider the implications of the T-bond futures having eligible for delivery a number of different T-bond issues. Because the CBT's system of conversion does not work exactly as it should, one of the eligible T-bonds winds up being cheapest to deliver, and the cheapest to deliver bond may change through time. We examine how this contract idiosyncracy affects the structure of the net cost of carry relation. Second, the valuation results of Chapters 7 through 9 were based on the assumption that the underlying asset price was log-normally distributed at the option's expiration. While this assumption may be reasonable for options on long-term bonds, it becomes less and less palatable as the term to maturity of the bond becomes short. A T-bill, for example, cannot have a price that exceeds its par value at expiration. To circumvent this problem, we assume interest rates are log-normally distributed at the option's expiration.

Net Cost of Carry Relation for CBT's T-Bond Futures

Table 17.9 summarizes the no-arbitrage price relations for futures and options written on coupon-bearing bonds. The relations are no different than they were for derivatives on stocks and stocks indexes. In place of the present value of cash dividends, we use the present value of coupons paid during the life of the derivative contract. The cost of carry relation of a bond futures is shown in Table 17.9 as being

$$F = Be^{rT} - FVC \tag{17.2}$$

where F is the futures price, B is the underlying coupon-bearing bond price, r is the continuously compounded risk-free interest rate, T is the time to expiration of the futures, and FVC is the future value of the coupons paid on the bond (if any) during the futures' life. What (17.2) says is that we should be indifferent between (1) buying a coupon-bearing bond futures contract for delivery of the bond at time T, and (2) buying the coupon-bearing bond and carrying it with short-term, risk-free borrowings to time T. Neither investment alternative involves a cash outlay today. Yet, both alternatives provide the bond at time T at a price known today. Note that FVC is subtracted from the future value of the bond investment on the right-hand side of (17.2) since any coupons paid prior to T are not included in the value of the bond at time T.

The net cost of carry relation (17.2) assumes that there is a single asset underlying the futures contract. Such is not the case for the CBT's T-bond futures contract—a number of bonds are eligible for delivery. Since the CBT's T-bond futures is the most active long-term interest rate futures traded in the United States, it is important that we develop an understanding of the cost of carry relation when multiple assets are eligible for delivery.

Before turning to the CBT's T-bond futures contract, it is helpful to be reminded about the bond price reporting conventions described in Chapter 2. In Chapter 2, we define B as the market price of a coupon-bearing bond, that is, the amount that we would pay if we decided to buy the bond. In market parlance, B is called the "gross price," the "full price," and/or the "dirty price" of the bond. What gets reported in the financial pages and displayed on pricing screens, however, is the "quoted price" or the "clean price," B^-. The quoted price B^- is the full bond price, B, less the accrued interest in the current coupon period, AI, that is, $B^- = S - AI$. Since we have to reconcile the T-bond futures pricing relation with observed market prices later in the chapter, it is important to recognize the reporting conventions upfront.

TABLE 17.9 Summary of no-arbitrage price relations and valuation equations/ methods for derivatives on interest rate products. Valuation equations/ methods are based on the assumption that bond/futures prices are log-normally distributed.

Arbitrage Relations		
Forward/Futures	$f = F = Be^{rT} - FVC$ or $fe^{-rT} = Fe^{-rT} = B - PVC$ where $FVC = \sum_{i=1}^{n} C_i e^{(T-t_i)}$ and $PVC = e^{-rT}FVC$	
European-Style:	**Options**	**Futures Options**
Lower bound for call	$c \geq \max(0, B - PVC - Xe^{-rT})$	$c \geq \max[0, e^{-rT}(F - X)]$
Lower bound for put	$p \geq \max(0, Xe^{-rT} - B + PVC)$	$p \geq \max[0, e^{-rT}(X - F)]$
Put-call parity	$c - p = B - PVC - Xe^{-rT}$	$c - p = e^{-rT}(F - X)$
American-Style:		
Lower bound for call	$C \geq \max(0, B - PVC - Xe^{-rT}, B - X)$	$C \geq \max(0, F - X)$
Lower bound for put	$P \geq \max(0, Xe^{-rT} - B + PVC, X - B)$	$P \geq \max(0, X - F)$
Put-call parity	$B - PVC - X \leq C - P \leq B - Xe^{-rT}$	$Fe^{-rT} - X \leq C - P \leq F - Xe^{-rT}$
Valuation Equations/Methods		
European-Style:	**Options**	**Futures Options**
Call value	$c = B^x N(d_1) - Xe^{-rT}N(d_2)$	$c = e^{-rT}[FN(d_1) - XN(d_2)]$
Put value	$p = Xe^{-rT}N(-d_2) - B^x N(-d_1)$	$p = e^{-rT}[XN(-d_2) - FN(-d_1)]$
	where $B^x = B - PVC$ $d_1 = \frac{\ln(B^x / Xe^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$	where $d_1 = \frac{\ln(F/X) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$
American-Style:		
Call and put values	Numerical valuation: binomial and trinomial methods.	Numerical valuation: quadratic approximation, and binomial and trinomial methods.

Cheapest-to-Deliver at Futures Expiration To see how the CBT's system *should* work, suppose we are standing on the expiration day of the futures contract T and consider the profit from selling the futures and buying and delivering eligible bond i. In the absence of costless arbitrage opportunities, all bonds should have a realized profit of 0, that is,

$$\pi_{i,T} = F_T(CF_i) + AI_{i,T} - B^-_{i,T} - AI_{i,T} = 0 \tag{17.3}$$

for all eligible bonds, where $F_T(CF_i) + AI_{i,T}$ is the invoice price received from delivering bond i and $B^-_{i,T} + AI_{i,T}$ is the price paid for the purchase of bond i. In practice, however, each bond will have a different value of $\pi_{i,T}$ because the CBT's system of conversion factors for the T-bond futures is not exact. All of the values of $\pi_{i,T}$ will be less than or equal to zero. The bond with the highest value of $\pi_{i,T}$ is called the "cheapest to deliver," and its $\pi_{i,T}$ will be equal to zero. The other bonds are said to be "more expensive to deliver" because the proceeds from the sale of the bond, $F_T(CF_i) + AI_{i,T}$, are less than the price paid for the bond that we are delivering, $B^-_{i,T} + AI_{i,T}$. Instinctively, we might think that an arbitrage profit is possible by reversing the trades (i.e., buying the futures and selling the bond) when $\pi_{i,T} < 0$. That intuition does not apply here since it is the individual who is short the futures that has the right to decide which bond to deliver. The individual who is long the futures contract has no say. None of the bonds will have $\pi_{i,T} > 0$ because that would imply that a costless arbitrage profit could be earned by buying the bond, selling the futures, and then delivering the bond on the futures commitment.

The technical reason why a single T-bond will be cheapest to deliver bond is that the CBT's conversion factors are computed assuming the zero-coupon yield curve is a flat 6% (i.e., $y = 0.06$ in (17.1)). Such a valuation procedure implicitly assumes that all coupon payments are reinvested as they are received at 6% yield until the end of the bond's life. If the current zero-coupon yield curve implies that the reinvestment rates for coupon payments are higher than 6%, the CBT's conversion factor will be too low for high-coupon bonds relative to low coupon bonds, hence low coupon bonds will be the preferred bonds to deliver. On the other hand, if the current zero-coupon yield curve implies that the reinvestment rates for coupon payments are lower than 6%, high-coupon bonds will be delivered.

ILLUSTRATION 17.1 Identify cheapest to deliver bond at futures expiration.

Suppose that three bonds with 15 years to maturity are eligible for delivery on the CBT's T-bond futures contract. They have coupon rates of 3%, 6%, and 9%, respectively. The conversion factors of the bonds are 0.7060, 1.000, and 1.2940, respectively. Identify the cheapest to deliver bond assuming the zero-coupon yield curve is a flat 6%. Then, assume the current zero-coupon yield curve is a flat 8% and identify the cheapest to deliver bond. Explain the results.

6% yield curve. With a 6% yield curve, the value of the 3%, 6% and 9% coupon bonds are

$$B_{3\%} = \sum_{t=1}^{30} \frac{1.5}{1.03^t} + \frac{100}{1.03^{30}} = 70.60$$

$$B_{6\%} = \sum_{t=1}^{30} \frac{3.0}{1.03^t} + \frac{100}{1.03^{30}} = 100.00$$

and

$$B_{9\%} = \sum_{t=1}^{30} \frac{4.5}{1.03^t} + \frac{100}{1.03^{30}} = 129.40$$

The CBT's system of conversion attempts to put all of these bonds on an equal footing with respect to being delivered on its T-bond futures contract. To translate each of these coupon-bearing bonds to a 6% coupon bond, we divide their values by the CBT's conversion factors to identify their implied values had they had 6% coupon rates. The results are shown in the table that follows. At a flat 6% yield to maturity, we are indifferent about which of the three bonds to deliver. All bonds have implied values equal to 100. In this case, the CBT's conversion factors work precisely as they should.

Bond	Bond Value at Yield of 6%	CBT Conversion Factor	Implied Value if 6% Coupon
3%	70.60	0.7060	100.00
6%	100.00	1.0000	100.00
9%	129.40	1.2940	100.00

8% yield curve. With an 8% yield curve, the bond values and implied values are shown in the table below. The 3% coupon-bearing bond, for example, has a value of 56.77, that is,

$$B_{9\%} = \sum_{t=1}^{30} \frac{1.5}{1.04^t} + \frac{100}{1.04^{30}} = 56.77$$

and an implied value of 80.41, that is,

$$56.77/0.7060 = 80.41$$

The implied values of the 6% and 9% coupon bonds are 82.71 and 83.96. Obviously, we would prefer to deliver the 3% bonds, since they have the lowest value.

Bond	Bond Value at Yield of 6%	CBT Conversion Factor	Implied Value if 6% Coupon	Modified Conversion Factor	Modified Value if 6% Coupon
3%	56.77	0.7060	80.41	0.6864	82.708
6%	82.71	1.0000	82.71	1.0000	82.708
9%	108.65	1.2940	83.96	1.3136	82.708

The last two columns in the previous table identify the problem. When the yield to maturity was 6%, the value of the 3% bond relative to the value of the 6% bond was 70.60/100.00 = 0.7060, exactly equal to the CBT's conversion factor. When the yield rises to 8%, the values of the bonds change relative to the 6% issue. At an 8% yield, holding a 3% coupon bond is like holding only 0.6864 6% bonds, and, holding a 9% is like holding 1.3136 6% bonds. If these "modified conversion factors" were used, all bonds would again be put on an equal footing for delivery purposes.

Cheapest-to-Deliver Prior to Futures Expiration Before maturity, as at maturity, the futures price is based on the price of the cheapest to deliver, and the cheapest to deliver is determined by finding the bond with the highest "cash-and-carry" portfolio[6] profit $\pi_{i,T}$,

$$\pi_{i,T} = F_0(CF_i) + AI_{i,T} + C_{i,t}e^{r(T-t)} - (B_{i,0}^- + AI_{i,0})e^{rT} \tag{17.4}$$

Note that (17.3) and (17.4) differ in some subtle ways. First, the futures price has a time subscript 0 to indicate that we are talking about today's price. The term $F_0(CF_i) + AI_{i,T}$ is the invoice price of the bond or the cash proceeds that we will receive from the sale of bond i when the futures contract expires. The term $(B_{i,0}^- + AI_{i,0})e^{rT}$ is the price that we paid for the bond at time 0 carried forward until time T at the risk-free rate of interest r. Finally, assuming a coupon interest payment was made at time t before the futures expiration (i.e., $t < T$), we invest the coupon at the risk-free rate of interest until time T and its contribution to the cash-and-carry portfolio profit is $C_{i,t}e^{r(T-t)}$. If no coupon is paid on bond i before futures expiration, this term disappears. If we calculate the cash-and-carry profit for all bonds eligible for delivery, we will get an array of values less than 0. The cheapest-to-deliver bond is the one with the highest cash-and-carry profit, and its value will be near zero.

Net Cost of Carry Relation The cash-and-carry profit equation (17.4) allows us to specify the net cost of carry relation for the T-bond futures contract. With a single T-bond i eligible for delivery, the relation can be obtained by setting the cash-and-carry profit equation (17.4) equal to 0 and solving for F_0, that is,

$$F_0 = \frac{(B_{i,0}^- + AI_{i,0})e^{rT} - AI_{i,T} - C_{i,t}e^{r(T-t)}}{CF_i} \tag{17.5}$$

But, many T-bonds are eligible for delivery and (17.4) is less than 0, even for the T-bond that is currently cheapest to deliver since there is no assurance that this bond will also be cheapest to deliver when the futures contract expires on day T. Consequently, the net cost of carry relation for the T-bond futures contract must be expressed as the inequality,

$$F_0 < \frac{(B_{i,0}^- + AI_{i,0})e^{rT} - AI_{i,T} - C_{i,t}e^{r(T-t)}}{CF_i} \tag{17.6}$$

The net cost of carry relation may also be written in a manner that explicitly recognizes the value of the *quality option*. The term "quality option" arose in the context of grain futures contracts, which allow the individual who is short the futures to deliver one of a number of different grades of a particular grain. The CBT's corn futures contract, for example, calls for the delivery of No. 2 yel-

[6] In this context, a cash-and-carry portfolio refers to buying the underlying T-bond, financing its purchase at the risk-free rate of interest, and selling the T-bond futures contract.

low corn at par, but also permits the delivery of No. 1 yellow at a 1½ cent premium over the futures contract price, and No. 3 yellow at a 1½ cent discount below the contract price. Come delivery day, the individual who is short the futures will be naturally choose to deliver the grade that is "cheapest."

The same situation arises with the T-bond futures contract. While the short may have entered a cash-and-carry position when bond *i* was cheapest to deliver, he will deliver bond *j* at maturity if its conversion price is below bond *i*'s, thereby earning a profit equal to the difference between the two prices. Thus the net cost of carry relation is

$$F_0 = \frac{(B_{i,0}^{-} + AI_{i,0})e^{rT} - AI_{i,T} - C_{i,t}e^{r(T-t)}}{CF_i} - \text{Quality option}_i \tag{17.7}$$

where bond *i* is the current cheapest to deliver.[7]

ILLUSTRATION 17.2 Value quality option embedded in T-bond futures.

Suppose that there exist two bonds that are eligible for delivery on the T-bonds futures contract. Bond A has a 6% coupon rate, one month remaining until the next coupon payment is made, a full price of 95, a conversion factor of 1.0000, and a volatility rate of 12%. Bond B has an 8% coupon rate, four months remaining until the next coupon payment is made, a full price of 107, a conversion factor of 1.2311, and a volatility rate of 15%. The correlation between rates of return of the two bonds is 0.9. The T-bond futures has a time to expiration of three months, and the risk-free interest rate is 4%. Find the cheapest-to-deliver bond as well as the value of the quality option when the T-bond futures is priced off the cheapest-to-deliver.

To identify the cheapest-to-deliver bond, we need to compute the implied futures price for each bond the first term on the right-hand side of (17.7). Bond A has an implied futures price of

$$IF_A = \frac{95e^{0.04(0.25)} - (6/2)e^{0.04(0.25-0.08333)}}{1.0000} = 92.935$$

Bond B has an implied futures price of

$$IF_B = \frac{107e^{0.04(0.25)}}{1.2311} = 87.874$$

Since bond B has the lowest implied futures price, it is currently the cheapest to deliver.

The individual who is short the futures contract currently plans to deliver the 8% coupon bond at the futures' expiration. There is some possibility, however, that the 6% coupon bond will become cheapest to deliver by the end of three months. If it does, the short futures will deliver the 6% bond. This "right-to-switch" or "quality option" can be valued using the exchange option valuation formula (8.20) from Chapter 8. In this particular case, the quality option is a put option that allows the short futures to deliver the first bond instead of the second if its implied futures price is less at the futures expiration.

[7] The value of the quality option can be computed using the exchange option valuation framework of Margrabe (1978) model. Recall that we applied the same framework in valuing call options with an indexed exercise price in Chapter 8.

To value this option, we need to recognize the fact that the value of a call option to buy asset 1 with asset 2 equals the value of a put option to sell asset 2 for asset 1. Under the first case, the option holder exercises the call if the price of asset 1 exceeds asset 2 at expiration. Under the second case, the option holder exercises the put if the price of asset 2 is below the price of asset 1 at expiration. The value of the quality option is therefore

$$p = 87.784e^{-0.04(0.25)}N(d_1) - 92.935e^{-0.04(0.25)}N(d_2) = 0.055$$

where

$$d_1 = \frac{\ln(87.784/92.935) + 0.5(0.0671^2)0.25}{0.0671\sqrt{0.25}}, \; d_2 = d_1 - 0.0671\sqrt{0.25}$$

$$\sqrt{0.12^2 + 0.15^2 - 2(0.90)(0.12)(0.15)} = 0.0671$$

This value can be verified using the OPTVAL Library function,

OV_FOPTION_VALUE_EXCHANGE(92.935,87.784,0.25,0.04,0.12,0.15,0.9, "p") = 0.055

The short futures also has a *timing option* that gives a choice about when during the contract month to deliver. The most valuable element in the timing option is called the *wildcard option*. In the delivery month the futures price at which delivery is made is the settlement price established at 2:00 PM when the market closes. The short has until 8:00 PM to declare delivery. Obviously, if news arrives that justifies a decline in bond prices, the short will choose to make delivery at the already established settlement price.

Repurchase Agreements[8] In discussing the net cost of carry relation for the T-bond futures contract, we used a risk-free rate of interest *r*. For T-bonds, it is important to digress and describe the most common form of financing the purchase of T-bonds, that is, a *sale repurchase agreement* (also known as a *repurchase agreement* or, simply, a *repo*).[9] A repo agreement is a single transaction with two separate trade confirmations: the first is the *sale* of the bond for immediate settlement, and the second is the *repurchase* of the bond for settlement at some future date. The repurchase price is known at the time the agreement is entered, and the difference between the repurchase price and the sales price is the interest on the loan. Specifically,

[8] More detailed discussions of repurchase and reverse repurchase agreements are provided in Stigum (1990, pp. 576-79), Fabozzi (1996, 131–135), and Tuckman (2002, pp. 303–10).

[9] Originally, just after World War II, repurchase agreements were used only for the purchase and sale of highly creditworthy and liquid debt securities such as U.S. Treasuries. Over time, the range of credits expanded to include collateral such as agency bonds and even investment grade corporate bonds. By the mid- to late 1990s, even speculative corporate bonds were being reversed out by hedge funds. The practice of reversing out "junk" bonds came to a screeching halt in the aftermath of the collapse of Long-Term Capital Management and the Asian debt crisis in 1997.

$$\text{Repurchase price} = \text{Sales price} \times \left[1 + \text{Repo rate}\left(\frac{n}{360}\right)\right] \quad (17.8)$$

where n is the number of days the repo is outstanding.[10] Thus the *repo rate* is the implicit rate of interest paid by the borrower to the lender. A repo with a term of one day is called an *overnight repo* and carries an interest rate called the *overnight repo rate*. Repos (rates on repos) with more than one day to maturity are called *term repos* (*term repo rates*). Repo rates are negotiated on a transaction by transaction basis and vary depending on such factors as the term of the repo and the credit-quality of the underlying collateral.

Why would a borrower choose to finance the purchase of the bond with a repo agreement rather than simply borrowing the money from the bank? The answer is that it is cheaper. From an economic perspective, a repurchase agreement is a collateralized loan.[11] The borrower posts his bond to the lender as collateral during the life of the agreement, and, in the event of default, the lender has the right to immediately sell the bond in the marketplace to cover his losses. By attaching specific collateral to the loan, the borrower garners a lower rate of interest. Why would a lender choose to enter a repo agreement rather than buy T-bills or money market instruments? The answer is that repo rates are generally higher than comparable term instruments and yet remain highly liquid, secured investments.[12]

The repo market carries with it a goodly amount of Wall Street jargon. In the interest of completeness, we will consider some of it. While the borrower is said to enter a *repurchase agreement,* the lender is said to enter a *reverse repurchase agreement* (also known as a *reverse repo* or, simply, a *reverse*). By lending his securities to provide collateral for the loan, the borrower is said to be *reversing out* securities or *selling collateral*. On the other hand, in accepting the collateral on the loan, the lender is said to be *reversing in* securities or *buying collateral*. The borrower is said *to repo securities*; the lender is said *to do a repo*.

Despite the fact that the collateral underlying repo agreements is generally high quality, repos are carefully structured to reduce credit risk. One way of controlling credit risk is to apply a *haircut* to the purchase price of the security to

[10] Like T-bills and Eurodollar time deposits, repo rate quotes adopt the banker's convention of a 360-day year

[11] The term *collateralized loan* is not meant to have any legal interpretation. Indeed, it is unclear whether a repo agreement is collateralized borrowing or a sequence of securities trades. If it were collateralized borrowing, the lender's right to sell the borrower's collateral immediately in the event of default may be restricted to protect the borrower's other creditors.

[12] Municipalities are frequent lenders in the repo market to manage their cash flows. Tax revenue is collected only periodically during the year. At the time of tax collection, the municipality has no immediate need for the cash. Repo agreements offer the municipality the ability to earn interest at competitive short-term rates with the safety of being secured loans until the cash needs to be disbursed. In addition, although the Federal Reserve removed interest rate ceilings on term deposits at commercial banks on March 31, 1986, it maintained the requirement that no interest be paid on demand deposits. (For a historical recount of the phase-out of Regulation Q, see Gilbert (1986).) Overnight repos are a convenient way to earn interest on what amounts to a demand deposit.

protect the lender from adverse market movements. A 5% haircut means that only 95% of the price of the bond is borrowed, with the bond held as collateral. The size of the haircut (i.e., the *amount of the margin*) will depend on the level of creditworthiness of the borrower, as well as the price risk, default risk, and liquidity of the collateral. Another way to control credit risk for term repos is to *mark-to-market* the collateral on a periodic (e.g., daily) basis. Suppose that a bond dealer has a haircut provision of 5% and securities with a market value of $100 million. By reversing out securities in the repo market, he can effectively borrow $95 million. Now, suppose that the market value of the securities drops to $99 million on the next day. Clearly, the lender is in a more precarious position given that the worth of the collateral has fallen. When this happens, the repo agreement may specify that there will be a *margin call*, in which case the borrower will be required to post additional $1 million in market value of collateral to bring the level back up to $100 million. Alternatively, the repo agreement may *reprice* the repo, in which case the principal amount of the repo is reduced from $100 million to $99 million and the borrower pays the lender $950,000 (i.e., 95% of $1 million). For ease of exposition, we ignore haircuts and the marketing-to-market features that may appear in repurchase agreements in this section. We also transform the repo rate to a continuously-compounded rate of interest, that is,

$$r = \frac{\ln\left[1 + \text{Repo rate}\left(\frac{n}{360}\right)\right]}{(n/365)}$$

To perform this computation, the OPTVAL library contains the function,

OV_IR_CONV_REPO_YIELD(*rate, days*)

where *rate* is the repo rate, and *days* is the term of the repo agreement in days.

Duration of the CBT's T-Bond Futures The duration of a futures contract is closely tied to the duration of the cheapest-to-deliver T-bond. To develop a formal relation, assume, for simplicity, that no coupons are paid during the futures' life and that the value of the quality option equals zero. Under these simplifying assumptions, the net cost-of-carry relation (17.7) may be rewritten as

$$F = \frac{Be^{rT}}{CF} \tag{17.9}$$

where B and CF are the price and the conversion factor of the cheapest-to-deliver bond. The change in the futures price with respect to a change in the level of interest rates r is

$$\frac{dF}{dr} = \frac{dB}{dr}\frac{e^{rT}}{CF} + \frac{BTe^{rT}}{CF} \tag{17.10}$$

Dividing the left-hand side by F and the right-hand side by Be^{rT}/CF, and simplifying, we get

$$\frac{dF/F}{dr} = \frac{dB/B}{dr} + T \tag{17.11}$$

In other words, the duration of the futures equals the duration of the cheapest-to-deliver bonds plus the term to maturity of the futures. The futures provides the underlying bond, but with deferred delivery.

Option Valuation Equations Under Log-Normal Bond Prices

The valuation methods derived in Chapters 7 through 9 assume that the asset underlying the option has a log-normal price distribution at the option's expiration. For options on long-term bonds or long-term bond futures, the assumption is reasonable. Such is the case for many traders in the CBT's T-bond futures option pit, who use the BSM model to make markets in options.[13]

ILLUSTRATION 17.3 Compute implied volatility from T-bond futures option price.

The CBT's options on T-bond futures are American-style, and expire the Friday, preceding by at least two business days, the last business day of the month preceding the contract month. (See Table 17.9.) At the close on Friday, October 11, 2002, the December 2002 futures price was 112-10, and the December 2002 options on the December 2002 futures have prices as follows:

Exercise Prices	Prices (in 64ths)	
	Call	Put
110	3-26	1-06
111	2-49	1-29
112	2-13	1-57
113	1-46	2-26
114	1-20	3-00
115	0-63	3-43

The risk-free interest rate is 1.772%. Compute the implied volatility for each reported option price using the quadratic approximation. Comment on the nature of the "implied volatility smile."

The first step is to translate prices to decimal form. The futures price is reported in (32nds), so 112-10 becomes 112.3125. The option prices are reported in 64ths, so the values shown in the previous table are:

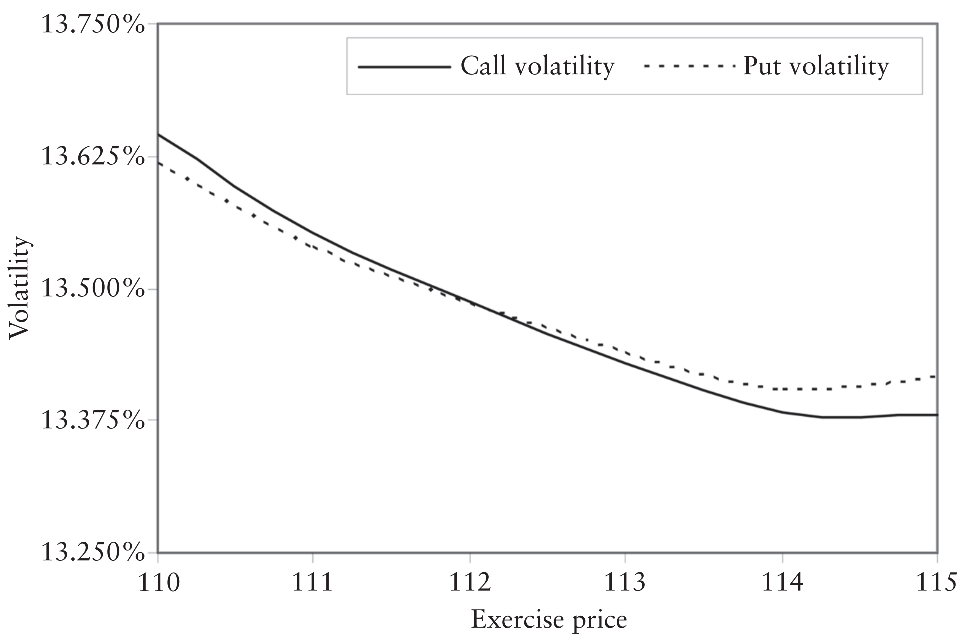
[13] Recall that the BSM formula reduces to the Black (1976) formula for valuating European-style futures options. A popular way of valuing American-style futures options is the quadratic approximation provided in Whaley (1986).

Exercise Prices	Prices (in 64ths)		Prices (in decimal)	
	Call	Put	Call	Put
110	3-26	1-06	3.406250	1.093750
111	2-49	1-29	2.765625	1.453125
112	2-13	1-57	2.203125	1.890625
113	1-46	2-26	1.718750	2.406250
114	1-20	3-00	1.312500	3.000000
115	0-63	3-43	0.984375	3.671875

The next step is to deduce the number of days to expiration. The last Friday, preceding by at least two business days the last business day of the month preceding the contract month is November 22, 2002, and the number of days to expiration is therefore 42.

Finally, we use the OV_OPTION_ISD function to compute the implied volatilities.[14] The results are as shown in the following table and figure. Note that the implied volatilities for calls and puts are approximately equal and that implied volatilities decrease modestly with exercise price.

Exercise Prices	Prices (in 64ths)		Prices (in decimal)		Implied Volatility	
	Call	Put	Call	Put	Call	Put
110	3-26	1-06	3.406250	1.093750	0.1365	0.1362
111	2-49	1-29	2.765625	1.453125	0.1355	0.1354
112	2-13	1-57	2.203125	1.890625	0.1349	0.1348
113	1-46	2-26	1.718750	2.406250	0.1343	0.1344
114	1-20	3-00	1.312500	3.000000	0.1338	0.1340
115	0-63	3-43	0.984375	3.671875	0.1338	0.1341



[14] OV_OPTION_ISD uses the quadratic approximation for American-style options.

Option Valuation Equations Under Log-Normal Interest Rates

For most assets, the assumption that the underlying asset has a lognormal price distribution at the option's expiration works well. For options on short-term debt instruments such as T-bill or Eurodollar futures, it does not. The log-normal price distribution is inappropriate because it allows prices to become infinitely high. T-bills and Eurodollar futures prices cannot exceed 100. To circumvent this problem, we assume that the yield, rather than the price, of the short-term debt instrument is log-normally distributed at the option's expiration. Under this assumption, the yield can fall to zero, in which case the market price of the short-term debt instrument becomes its predetermined maturity value. On the other hand, if the yield rises without limit, the market price of the debt instrument converges to zero.

To illustrate this option valuation approach, we focus on the CME's Eurodollar futures options. Although these options are American-style, we will assume that they are European-style for expositional convenience. Aside from the assumption that the forward Eurodollar rate is log-normally distributed at the end of the option's life, we invoke the same assumptions that we used in Chapters 7 through 9. In particular, since both the futures and futures options are actively traded, we adopt risk-neutral valuation.

Under risk-neutral valuation, the current value of a European-style Eurodollar futures option today is the present value of the expected future terminal value, that is,

$$c = e^{-rT}E(\tilde{c}_T) \tag{17.12}$$

where the expected terminal value is discounted to the present at the zero-coupon rate corresponding to the expiration of the option. The terminal value of the option is, in turn, a function of the Eurodollar futures index level, F_T, that is,

$$\tilde{c}_T = \begin{cases} \tilde{F}_T - X & \text{if } F_T > X \\ 0 & \text{if } F_T > X \end{cases} \tag{17.13}$$

If the terminal futures price is log-normally distributed, we would evaluate $E(\tilde{c}_T)$ in the same manner as we did in Chapter 7. With the forward Eurodollar rate, R, being log-normally distributed, however, we must rewrite the option's payoff function as

$$\tilde{c}_T = \begin{cases} (100 - X) - \tilde{R}_T & \text{if } R_T < 100 - X \\ 0 & \text{if } R_T \geq 100 - X \end{cases} \tag{17.14}$$

where we have merely substituted the fact that the Eurodollar futures price is an index level created by subtracting the Eurodollar rate from 100, that is, $F = 100 - R$. But, equation (17.14) looks surprisingly familiar. It is the terminal value function of a European put option where $\tilde{R}_T$ has replaced $\tilde{S}_T$ and where $100 - X$ has replaced

X. Since R_T is log-normally distributed, the BSM European-style put option formula from Chapter 5 can be applied directly. The expected terminal call price is

$$E(\tilde{c}_T) = (100 - X)N(-d_2) - (100 - F)N(-d_1) \tag{17.15}$$

where

$$d_1 = \frac{\ln[(100 - F)/(100 - X)] + 0.5\sigma_R^2 T}{\sigma_R\sqrt{T}} \text{ and } d_2 = d_1 - \sigma_R\sqrt{T}$$

where σ_R is the standard deviation of the logarithm of the yield ratios, $\ln(R_t/R_{t-1})$. Substituting (17.15) into (17.12), we find that the value of a European-style call option on a Eurodollar futures contract is

$$c = e^{-rT}[(100 - X)N(-d_2) - (100 - F)N(-d_1)] \tag{17.16}$$

By virtue of put-call parity for European-style futures options, the value of a European-style put option on a Eurodollar futures contract is

$$p = e^{-rT}[(100 - F)N(-d_1) - (100 - X)N(-d_2)] \tag{17.17}$$

ILLUSTRATION 17.4 Compute implied volatility from Eurodollar futures option price.

The CME's options on Eurodollar futures are American-style, and expire the second London business day before the third Wednesday of the contract month. At the close on Friday, October 11, 2002, the December 2002 futures price was 98.30, and the December 2002 options on the December 2002 futures have prices as follows:

	Prices (in decimal)	
Exercise Prices	Call	Put
9775	0.5575	0.0075
9800	0.3175	0.0175
9825	0.1450	0.0950
9850	0.0650	0.2650
9875	0.0250	0.4750
9900	0.0125	0.7100

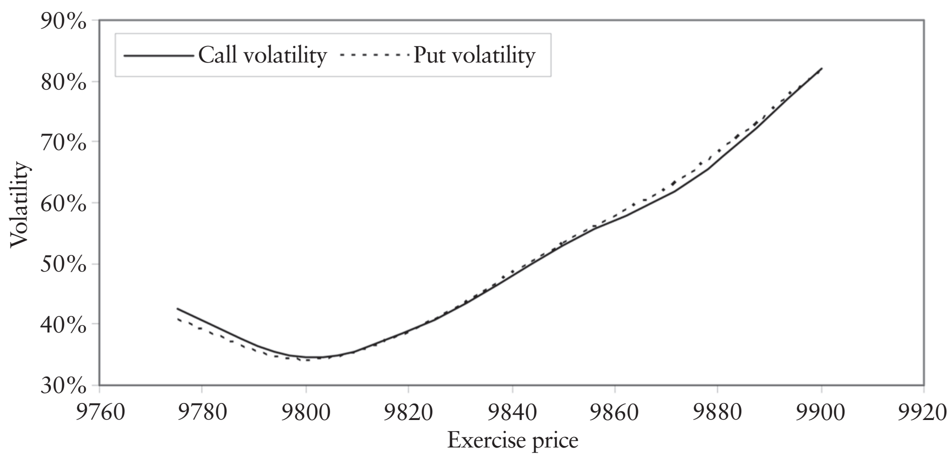
The risk-free interest rate is 1.771%. Compute the implied volatility for each reported option price using the European-style option valuation formula. Comment on the nature of the "implied volatility smile."

The first step in this illustration is to deduce the number of days to expiration. The second London business day before the third Wednesday of the contract month is December 16, 2002, and the number of days to expiration is therefore 66.

Using the OV_OPTION_ISD function to compute the implied volatilities, we find:

	Prices (in decimal)		Implied Volatility	
Exercise Prices	Call	Put	Call	Put
9775	0.5575	0.0075	0.4260	0.4074
9800	0.3175	0.0175	0.3466	0.3409
9825	0.1450	0.0950	0.4065	0.4059
9850	0.0650	0.2650	0.5309	0.5337
9875	0.0250	0.4750	0.6368	0.6479
9900	0.0125	0.7100	0.8200	0.8162

As the figure that follows shows, the implied volatilities for calls and puts are approximately the same even though the options are American-style. At very low exercise prices, however, there is a slight difference. At these exercise prices, the call is in the money, and the implied volatility is higher because the early exercise premium of the option is being ignored. Interestingly, a similar pattern does not appear for puts at high exercise prices.



RISK MANAGEMENT APPLICATIONS

This section focuses on some straightforward interest rate risk management problems using exchange-traded interest rate products. Many other interest rate strategies are discussed in Chapters 18 and 19.

Short-Term, Long Hedge

Interest rate futures can be used to lock in forward interest rates. Suppose, for example, that on August 31, 2000 a company anticipates a cash inflow of $5,000,000 the following September 18, 2000. The cash, when it is received will be placed in a three-month certificate of deposit until December when it will be used to partially finance a major capital expenditure that the firm plans. Suppose also that the company's financial analyst expects short-term three-month CD rates to fall to a level of 5.5% by December, while the current implied three-month forward rate of the September 2000 Eurodollar futures based on its reported price of 93.32 is 6.68%. What can the company do to lock in the higher rate of interest?

Buying and selling Eurodollar futures contracts are a cost-efficient means of locking-in forward rates of interest. The solution to this problem is to buy five September 2000 Eurodollar futures at 93.32. To see how we have locked-in the 6.68% rate, consider what happens on September 18 when the Eurodollar futures expires. If the three-month rate is 5% at that time, the September 2000 futures will be priced at 95.00. That means we will have posted a gain of (9500 – 9332) × $25 = $21,000. We take this gain as well as the $5 million cash payment and deposit them at the 5% interest rate. At the end of three months, the terminal value of our deposit is

$$\$5{,}021{,}000[1 + 0.05(91/360)] = \$5{,}084{,}459.86$$

Thus the simple rate of return on the $5 million cash flow over the three-month period is

$$\$5{,}084{,}459.86/\$5{,}000{,}000 = 1.6892\%$$

and the nominal interest rate on an annualized basis is

$$1.6892\%\left(\frac{360}{91}\right) \approx 6.68\%$$

as promised.

Long-Term, Short Hedge

In Chapter 2, we discussed hedging long-term interest rate risk exposure using duration-based techniques. We now modify these techniques to use futures contracts as the hedge instrument. Hedging means finding the number of futures to buy or sell such that the value of the overall hedged portfolio does not change if interest rates change, that is,

$$\Delta B_P + n_F \Delta F = 0$$

where ΔB_P and ΔF are the changes in value of your bond position and the futures resulting from a change in interest rates, Δy. Duration-based hedging means approximating the change the changes of value with the product of duration and bond value, that is,

$$D_P B_P + n_F D_F F = 0$$

where D_P and D_F are durations of the bond portfolio and the futures contract, respectively. The number of units of the hedge instrument to buy or sell is therefore given by

$$n_F = -\frac{D_P B_P}{D_F B_F}$$

ILLUSTRATION 17.5 Short hedge bond portfolio with long-term interest rate risk.

Suppose we currently manage a \$50 million bond portfolio with a duration of 10.00. Suppose also that the T-bond futures contract has a duration of 12.50 and a price of 99⁷⁄₃₂. Find the futures hedge that completely negates the long-term interest rate risk exposure.

The optimal number of futures contracts to sell is

$$n_F = -\frac{10 \times 50{,}000{,}000}{12.50 \times 99.21875 \times 1{,}000} = -411.44$$

Once the hedge is in place, the combination of long bonds and short futures should behave as if it were \$50 million invested in T-bills.

Equivalence of Duration-Based and OLS Regression Approaches The duration-based approach to hedging formula derived above shows the optimal number of futures to sell against a long position in bonds is

$$h^* = -\frac{D_B B}{D_F F}$$

Yet, in the minimum variance hedging discussion of Chapter 5, we argued that the optimal hedge ratio is $-\alpha_1$ in an OLS regression of the changes in bond portfolio value on the changes in the value of the T-bond futures, that is,

$$\Delta B = \alpha_0 + \alpha_1 \Delta F + \varepsilon$$

Can these seemingly disparate results be reconciled?

To understand that these results are essentially the same, rewrite the duration-based optimal hedge ratio as follows:

$$h^* = -\frac{D_B B}{D_F F} = \frac{-\dfrac{\Delta B / B}{\Delta y} B}{\dfrac{\Delta F / F}{\Delta y} F} = -\frac{\Delta B}{\Delta F}$$

where the slope coefficient in the regression is also

$$\alpha_1 = \frac{\Delta B}{\Delta F}$$

Thus from an analytical perspective, the results are the same. There will be slight differences in implementation, however, since they use different sources of information.

Asset Allocation

The *asset allocation* decision refers to the allocation of fund wealth among various asset categories including stocks, bonds (government and corporate), real estate, and so on. Deep and liquid futures markets on the different asset categories provide cost-efficient vehicles for altering temporarily the asset mix or helping unwind or create large asset positions without incurring significant market impact costs.

ILLUSTRATION 17.6 Adjust asset allocation using futures.

Suppose that we currently manage a $100 million portfolio consisting of $50 million in stocks and $50 million in long-term government bonds. The stock portfolio is well diversified and has a beta of 1.5. The bond portfolio has a duration of 12. Change the asset allocation of this portfolio from 50% stocks and 50% bonds to 100% stocks using T-bond futures and S&P 500 index futures. Assume the T-bond futures has a duration of 9 and a price of 96. Assume that current S&P 500 index level is 1,500.

First, we neutralize the long-term interest rate risk exposure. To do so, we sell T-bond futures, the exact number determined by

$$h_{TBF} = -\frac{50{,}000{,}000(12.00)}{9 \times 96.00 \times 1{,}000} = -694.44$$

By selling this number of futures, we eliminate the long-term interest rate risk exposure of the government bonds. What we have done, in essence, is transform the $50 million long-term government bond portfolio into $50 million in T-bills. Hence, as of this moment, the overall portfolio contains $50 million in cash and $50 million in stocks.

The next step is to create $50 million more in stock. We do this using the $50 million in cash and by buying S&P 500 index futures. The number of futures is given by

$$n_F^* = (\beta^* - \beta_P)\left(\frac{P}{S}\right)$$

where β_P, in this context, is the beta of the T-bills, β^* is the desired beta of the portfolio (i.e., 1.5), P is the desired investment in stocks, and S is the market value of one index unit (i.e., the index level times the futures denomination). The number of futures contracts to buy is therefore

$$n_F^* = 1.50\left(\frac{50{,}000{,}000}{1{,}500(250)}\right) = 200$$

The 200 S&P 500 futures together with the $50 million in T-bills creates a $50 million stock portfolio with a beta of 1.50. Together with the $50 million invested in a stock portfolio with a beta of 1.50, we now have $100 million invested in stocks.

SUMMARY

This chapter discusses exchange-traded interest rate products. Interest rate derivatives are by far the largest derivatives product category, although it may not seem so considering that most of the trading is conducted in the OTC market. The first section of this chapter reviews key contracts in exchange-traded markets. The second section deals with valuation. For the most part, the principles and valuation methods of Chapters 5 through 9 can be applied directly, with two notable exceptions. First, the no-arbitrage price relation for the CBT's T-bond futures must be modified to account for the fact that the seller has an option to deliver any one of a number of eligible bond issues. Second, for options on short-term debt instruments, the log-normal price distribution assumption is clearly inappropriate. The price of a T-bill, for example, can never exceed its par value. Consequently, a new methodology for valuing interest rate options is developed. We rely on an assumption that the short-term interest rate is log-normally distributed. Section three contains three important risk management applications using interest rate derivatives—a short-term long hedge, a long-term short hedge, and asset allocation.

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