CHAPTER 8

Valuing Nonstandard Options Analytically

In Chapter 7, we valued standard European-style options analytically with the Black-Scholes (1973)/Merton (1973) option valuation framework. This chapter continues to focus on options that can be valued analytically within the BSM framework, however, the types of options that we examine are unusual or nonstandard.[1] While we discuss eleven different types of contracts, do not be misled. There are a virtually limitless number of variations of derivative contracts that have or can be structured. Some can be valued analytically. These are the focus of this chapter. Some require the use of numerical methods. These are the focus of the next chapter. As we proceed through this chapter describing the different types of contracts and their analytical valuation equations, it is important to try to imagine possible applications. In many instances, the contracts have sensible return/risk management properties. In other instances, the contracts seem only to be a cleverly structured bet.

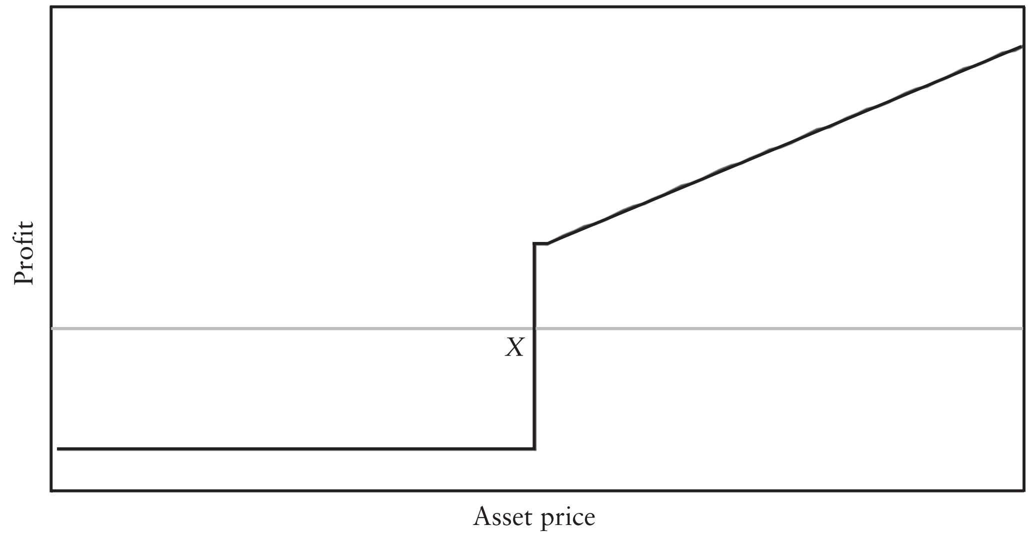
ALL-OR-NOTHING OPTIONS

In Chapter 5, we showed that the valuation equations for asset-or-nothing and cash-or-nothing call and put options were impounded within the BSM call and put formulas. Recall that an asset-or-nothing call that pays one unit of the asset at time T if the asset price exceeds the exercise price X. The terminal profit from buying an asset-or-nothing call option is shown in Figure 8.1. Note that, for terminal asset prices below X, the option holder forfeits the premium that he paid for the option at the outset. For terminal prices above X, the option holder receives one unit of the asset, which at least partially covers the original cost of the option. Under the BSM assumptions, the value of a European-style asset-or-nothing option is

$$c_{AON}(X, T) = e^{-iT}N(d_1) \tag{8.1}$$

[1] The label "exotic" has often been applied to nonstandard options.

FIGURE 8.1 Terminal profit from buying an asset-or-nothing call option.



where[2]

$$d_1 = \frac{\ln(Se^{-iT}/Xe^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}$$

The terminal profit from buying a cash-or-nothing call option is shown in Figure 8.2. For terminal asset prices below X, the option holder forfeits the original cash-or-nothing option premium. For terminal prices above X, the option holder receives one dollar. Under the BSM assumptions, the value of a European-style cash-or-nothing option is

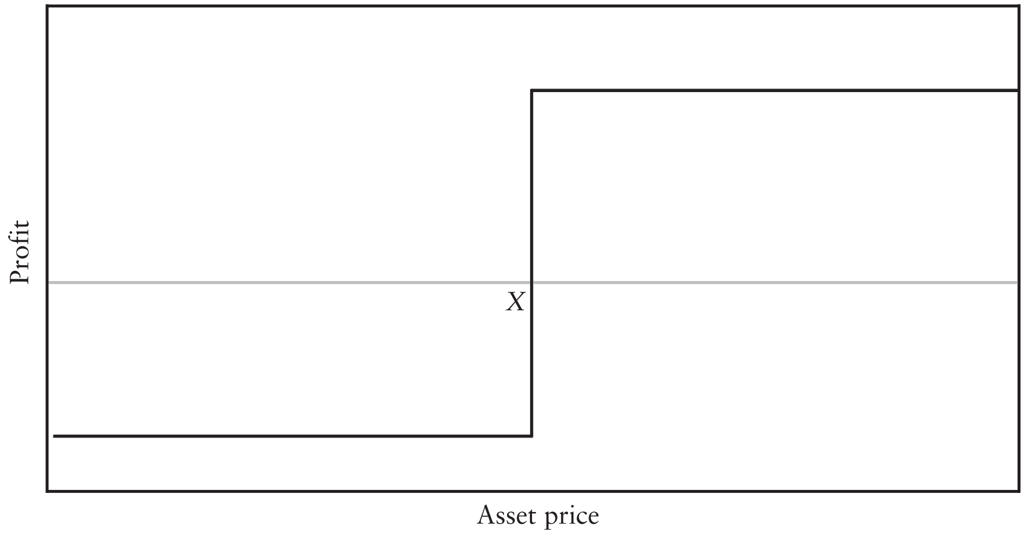
$$c_{CON}(X, T) = e^{-rT}N(d_2) \tag{8.2}$$

where the fixed cash amount equals one dollar.

A standard European-style call option provides the right to buy the underlying asset whose current price is S for amount of cash equal to X at time T. To construct a standard call option, we buy S units of an asset-or-nothing call and sell X units of a cash-or-nothing call. Thus the valuation-by-replication principle says that the value of a standard European-style call option is

[2] As noted in Chapter 7, there are a number of different, albeit equivalent, expressions for d_1 in European-style option valuation problems. The prepaid forward version is used here to minimize the number of redundant computations.

FIGURE 8.2 Terminal profit from buying a cash-or-nothing call option.



$$c_{BSM} = Se^{-iT}N(d_1) - Xe^{-rT}N(d_2) \tag{8.3}$$

In the interest of completeness, an asset-or-nothing put option pays one unit of the asset at time T if the asset price is below the exercise price and is valued as

$$p_{AON}(X, T) = e^{-iT}N(-d_1) \tag{8.4}$$

A cash-or-nothing put pays one dollar in cash at time T if the asset price is below the exercise price and is valued as

$$p_{CON}(X, T) = e^{-rT}N(-d_2) \tag{8.5}$$

A standard European-style put option provides the right to sell the underlying asset S for amount of cash equal to X at time T. To construct a standard put option, we sell S units of an asset-or-nothing put and buy X units of a cash-or-nothing put. Thus, the value of a standard European-style put option is

$$p_{BSM} = Xe^{-rT}N(-d_2) - Se^{-iT}N(-d_1) \tag{8.6}$$

ILLUSTRATION 8.1 Value cash-or-nothing call option.

Suppose your uncle tells you that he will give you \$100 if XYZ's stock price is greater than \$100 in six months time. XYZ's current stock price is \$90 a share, its dividend yield rate is 1%, and its volatility rate is 20%. The risk-free interest rate is 3%. What is the value of his gift to you?

First, compute the prepaid forward prices for the underlying asset and risk-free bonds, that is, $Se^{-iT} = 90e^{-0.01(0.5)} = 89.551$ and $Xe^{-rT} = 100e^{-0.03(0.5)} = 98.511$. Next, compute the integral limit

$$d_2 = \frac{\ln(89.551/98.511) + 0.5(0.20^2)0.5}{0.20\sqrt{0.5}} = -0.7450$$

Finally, plug the information into formula (8.2), that is,

$$\begin{aligned} c_{CON} &= e^{-0.03(0.5)}N(-0.7450) \times 100 \\ &= 24.453 \end{aligned}$$

The values of asset-or-nothing and cash-or-nothing options where the underlying asset has a current price of one dollar can be computed using the OPTVAL library function

OV_NS_ALL_OR_NOTHING_OPTION(*s, x, t, r, i, v, cp, ac*)

where *s* is the asset price, *x* is the exercise price, *t* is the time to expiration expressed in years, *r* is the risk-free interest rate, *i* is the income rate of the asset, *v* is the asset return volatility rate, *cp* is a (c)all/(p)ut indicator, and *ac* is an (a)sset/(c)ash indicator. The value of the option in this illustration is

OV_NS_ALL_OR_NOTHING_OPTION(90,100,0.5,0.04,0.01,0.20, "c", "c") = 0.24453

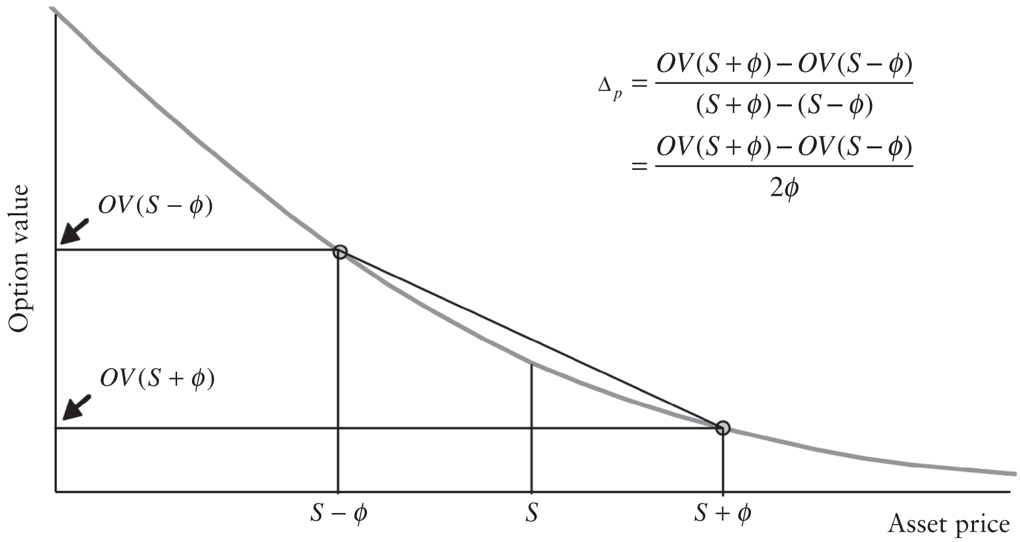
Measuring Risk of All-or-Nothing Options

All-or-nothing options are useful more as a pedagogic device than they are in practice. The reason is that they are expensive and difficult to hedge, particularly for short times to expiration. To address the hedging issue, let us consider the risk characteristics (i.e., the Greeks) of all-or-nothing options. Since we have analytical valuation formulas, we can compute analytically the Greeks of all-or-nothing options by taking the partial derivatives of the formulas with respect to each of the option formula's determinants (e.g., *S*, the asset price, σ, the volatility rate, and so on). But doing so necessarily involves developing more expressions in a chapter that will have no shortage of formulas.[3]

Instead, in this chapter, we measure risk characteristics numerically. The procedure is straightforward. Recall that the delta of an option is the change in option value with respect to a change in asset price. To obtain the delta of an option numerically, we can perturb the current asset price *S* by a small amount ϕ in either direction, that is, $S + \phi$ and $S - \phi$, and value the option at each asset price, $OV(S + \phi)$ and $OV(S - \phi)$. Figure 8.3 illustrates. The valuation function $OV(.)$ can be any of the valuation methodologies discussed in this chapter, the last chapter, or the next. The values generated in Figure 8.3 were generated using the BSM formula for a European-style put. While what we would like to

[3] It is also important to recognize that many OTC derivative contracts that we will discuss in later chapters have American-style option features or multiple, interrelated contingencies. In many, if not the majority, of these cases, analytical valuation is not possible and numerical methods must be applied. With no analytical formulas, the risk characteristics of these agreements must, necessarily, be computed numerically.

FIGURE 8.3 Numerical approximation for the delta of a put option.



measure is the slope of the OV function at the asset price S, we approximate the slope of the function by taking the ratio of the difference between the computed option values to the difference between the perturbed asset prices, that is,

$$\Delta_p = \frac{OV(S+\phi)-OV(S-\phi)}{(S+\phi)-(S-\phi)}$$

In general, all of the Greeks for options can be measured using the expression,

$$\text{Greek}_k = \frac{OV(k+\phi)-OV(k-\phi)}{2\phi} \tag{8.7}$$

where OV represents any valuation method that we use in this book, k is the option determinant of interest (e.g., S for delta risk, σ for vega risk, and so on), and ϕ is a small positive constant selected by the user. The gamma, that is, the change in the delta with respect to a change in the asset price, can be computed using

$$\text{Gamma} = \frac{OV(S+\phi) - 2\times OV(S+\phi) + OV(S-\phi)}{\phi^2} \tag{8.8}$$

ILLUSTRATION 8.2 Assess accuracy of risk measures computed numerically.

Consider a six-month European-style put option whose exercise price is 50. Assume the underlying asset has a price of 49, a dividend yield of 1%, and a volatility rate of 20%, compute the delta, gamma, and vega of the put analytically and then numerically. Assume the risk-free rate of interest is 3%.

First, compute the delta value of the European-style put analytically. From Chapter 5, we know that the delta value of the put is

$$\Delta_p = -e^{-0.01(0.5)}N(-d_1) = -0.4981$$

where

$$d_1 = \frac{\ln(49e^{-0.01(0.5)}/50e^{-0.03(0.5)}) + 0.5(0.20^2)(0.5)}{0.20\sqrt{0.5}} = -0.00143$$

and

$$N(0.00143) = 0.5006$$

Next compute the delta value of the put numerically by perturbing the asset price by, say, 0.1. Using the BSM put formula (8.6) for the function OV in (8.8), the numerical value of delta is

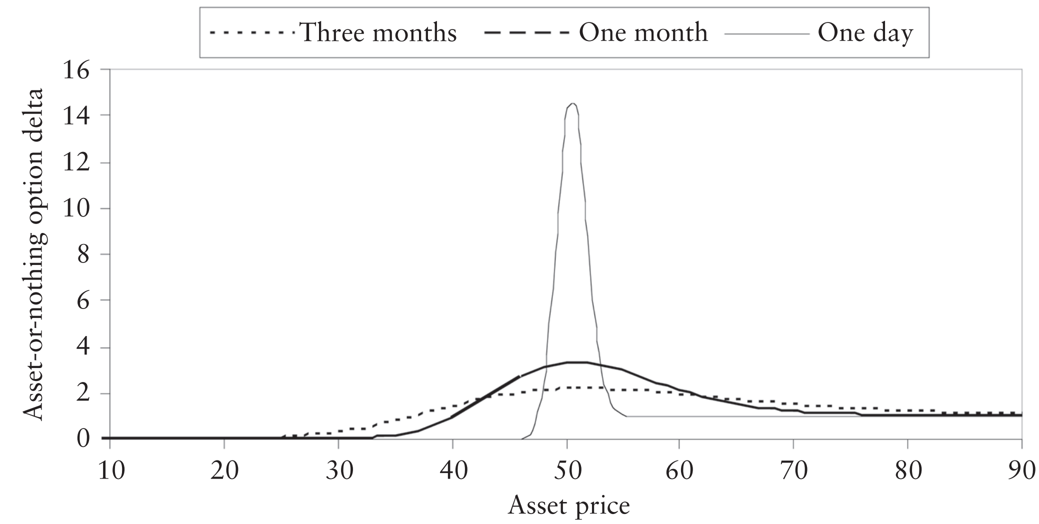
$$\begin{aligned}\Delta_p &= \frac{OV(40+0.1) - OV(40-0.1)}{(49+0.1)-(49-0.1)} \\ &= \frac{2.9702 - 3.0698}{0.2} = -0.4981\end{aligned}$$

In other words, the numerical (approximate) delta value of the put is accurate to four decimal places. The accuracy of the approximation is affected by the size of the perturbation parameter. To judge the appropriate size, experiment with an option whose Greeks are analytically tractable. Below is a summary of the results for delta, gamma, and vega of the put computed using (8.7) and (8.8). All of them are accurate to four decimal places for the perturbation amounts shown.

Greek	Analytical Value of Greek	Perturbation Amount, ϕ	Numerical Value of Greek
Delta	–0.4981	0.1	–0.4981
Gamma	0.0573	0.5	0.0573
Vega	13.7537	1.00%	13.7537

Returning to risk characteristics of all-or-nothing options that are the focus of this section, consider in Figure 8.4, which shows the distribution of delta values of asset-or-nothing call options with three months, one month, and one day to expiration as a function of the underlying asset price. These values were computed numerically using one dollar increments in the asset price. Several observations about Figure 8.4 are noteworthy. First, note that for deep out-of-the-money options, the deltas are near zero. The reason is simple. With virtually no chance of ever being in the money at expiration, the asset-or-nothing option is insensitive to movements in asset price. Second, note that deep in-the-money options have deltas near one. With virtually no chance of ever being out of the money, the option price behaves just like the asset price. Third, and perhaps most importantly, the delta value of at-the-money, asset-or-nothing call options (unlike standard call options) can be well in excess of one and increases as the

FIGURE 8.4 Delta values of asset-or-nothing call options with three months, one month, and one day to expiration. (X = 50, r = 0.05, i = 0.00, σ = 0.50).



time to expiration grows short. The maximum delta value is, or course, equal to the asset price. With only a few minutes to expiration, an asset price movement from slightly out of the money to slightly in the money will cause the asset-or-nothing option value to go from 0 to S.

GAP OPTIONS

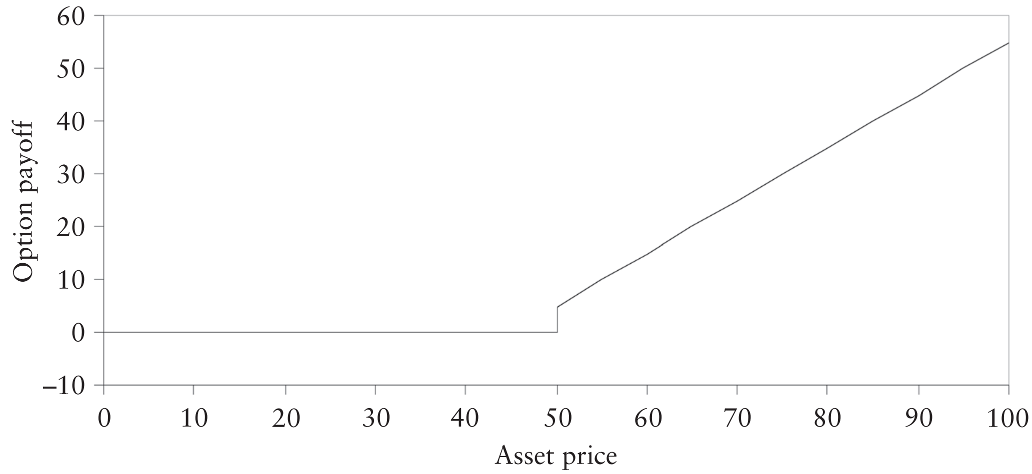
A *gap option* is an option whose payoff is determined by the exercise price X_1, but another constant X_2 determines whether or not the payoff is made. Consider a gap call option for example. Suppose X_1 = 45 and X_2 = 50. Figure 8.5 shows the call's payoff at expiration. Note that, over the asset price interval between 45 and 50, the call's payoff is 0. This is because the trigger price X_2 = 50 has not been reached. Once the asset price goes beyond the trigger price, the call's payoff is the difference between the asset price and X_1.

With the asset-or-nothing and cash-or-nothing valuation equations in hand, valuing a gap call option is a straightforward task.[4] Consider a portfolio is long an asset-or-nothing call with exercise (trigger) price X_2 and is short X_1 cash-or-nothing calls with exercise (trigger) price X_2. This portfolio has payoffs identical to those shown in Figure 8.5. In the absence of costless arbitrage opportunities, therefore, the value of a gap call option can be identified using (8.1) and (8.2), that is,

$$c_{gap}(X_1, X_2) = Se^{-iT}N(d_1) - X_1 e^{-rT}N(d_2) \tag{8.9}$$

[4] The construction of the gap option valuation formula from all-or-nothing options was first shown in Rubinstein and Reiner (1991b).

FIGURE 8.5 Terminal payoff of a gap call option with $X_1 = 45$ and $X_2 = 50$.



where

$$d_1 = \frac{\ln(Se^{-iT}/X_2e^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

and $d_2 = d_1 - \sigma\sqrt{T}$. The value of a gap put option is

$$p_{\text{gap}}(X_1, X_2) = X_1e^{-rT}N(-d_2) - Se^{-iT}N(-d_1) \tag{8.10}$$

It is important to note that there is no restriction on whether the trigger price X_2 is greater than or less than the exercise price X_1. Figure 8.5 shows the payoffs of a gap call under the condition $X_1 < X_2$. Where $X_1 > X_2$, however, we get the unusual payoff structure shown in Figure 8.6. Because the trigger price is reached before the exercise price, the call option holder is forced to exercise even though it is not profitable to do so. In the asset price interval between 50 and 55, the option holder pays $S - 55$. Indeed, there is exercise price at which the gap call option premium (8.6) will be equal to 0.

ILLUSTRATION 8.3 Value gap call option.

Compute the value of a six-month European-style gap call option whose exercise price is 55 and whose trigger price is 50. Assume the underlying index has a level of 49, a dividend yield of 1%, and a volatility rate of 20%. The risk-free rate of interest is 3%.

The values of the prepaid forward and exercise prices in the gap call option formula are:

$$Se^{-iT} = 49e^{-0.01(0.5)} = 48.756$$

$$X_1e^{-rT} = 55e^{-0.03(0.5)} = 54.181$$

$$X_2e^{-rT} = 50e^{-0.03(0.5)} = 49.256$$

The values of d_1 and d_2 are

$$d_1 = \frac{\ln(48.756/49.256) + 0.5(0.20^2)(0.5)}{0.20\sqrt{0.5}} = -0.0014$$

and $d_2 = -0.1429$. The value of the gap call is therefore

$$c_{\text{gap}}(55{,}50) = 48.756N(-0.0014) - 54.181N(54.181) = 0.3367$$

This value can also be obtained using the OPTVAL function

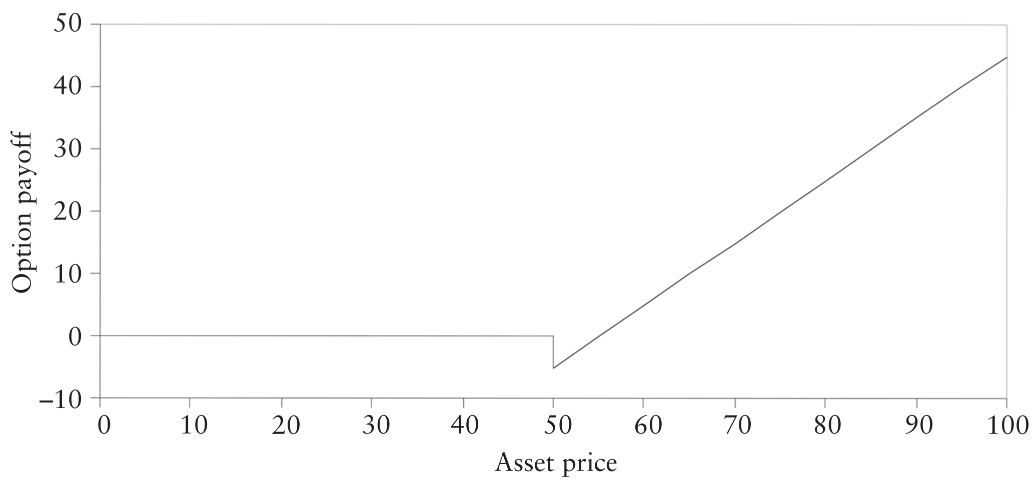
OV_NS_GAP_OPTION(*s, x1, x2, t, r, i, v, cp*)

where *x1* is the exercise price of the option and determines the payoff, *x2* is the price that triggers exercise, and all other function notation is as defined earlier in the chapter. Thus

OV_NS_GAP_OPTION(49, 55, 50, 0.5, 0.03, 0.01, 0.20, "c") = 0.3367

One final aspect of gap options is worthwhile noting. As the difference (i.e., gap) between the exercise price and the trigger price grows large, the gap call (put) option value decreases (increases). Indeed, where the exercise price is 55.77 rather than 55 in this illustration, the gap call value is 0. Figure 8.7 shows gap call and put option values for a range of exercise prices where all other parameter values are as described above.

FIGURE 8.6 Terminal payoff of a gap call option with $X_1 = 55$ and $X_2 = 50$.

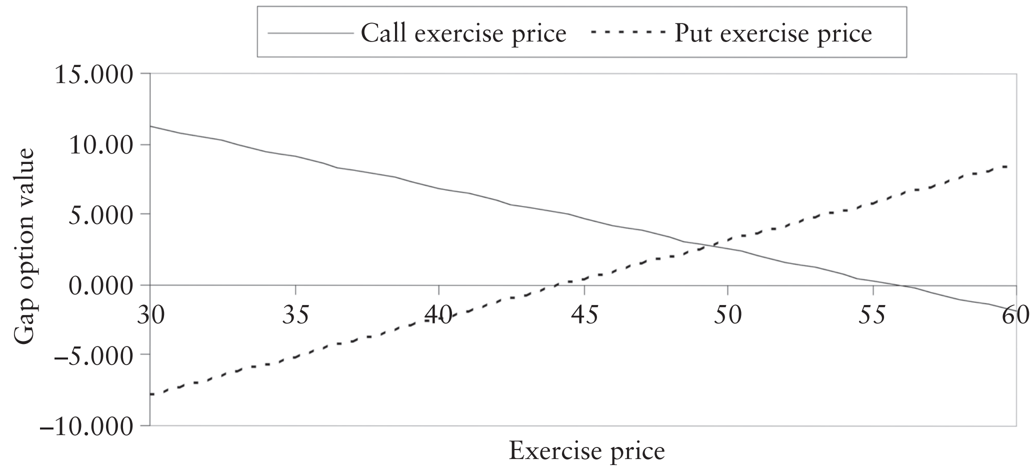


CONTINGENT PAY OPTIONS

A *contingent pay option* is an option whose premium is set today but is paid at expiration contingent upon the option being in the money.[5] Naturally, such an option will be more expensive that a standard European-style option, however, you will pay for the option only in the event the option is in the money at expira-

[5] Contingent pay options are also referred to as *pay-later options* or *collect-on-delivery options*.

FIGURE 8.7 Gap option values as a function of exercise price. Option parameters are: $S = 49$, $X_2 = 50$, $T = 0.5$, $r = 0.03$, $i = 0.01$, and $\sigma = 0.20$.



tion. Again, identifying the appropriate replicating portfolio is the key to solving for option value. To value a European-style contingent pay call option, consider the payoffs of a portfolio formed by buying a standard call option and selling a cash-or-nothing call with the cash amount being equal to the current value of the contingent pay option, $c_{\text{contingent pay}}$. At time T, the portfolio has a terminal value equal to (1) 0 if $S_T < X$ and (2) $S_T - X - c_{\text{contingent pay}}$ if $S_T \geq X$, exactly the required payoffs. The value of the portfolio at time 0 is $c_{BSM} - e^{-rT}N(d_2)c_{\text{contingent pay}}$, however, since this contract by its nature has no upfront premium, we must set the initial portfolio value equal to zero and solve for the contingent pay option premium. The value of a European-style contingent pay call option is

$$c_{\text{contingent pay}} = \frac{c_{BSM}}{e^{-rT}N(d_2)} = \frac{Se^{-iT}N(d_1) - X_1e^{-rT}N(d_2)}{e^{-rT}N(d_2)} \tag{8.11}$$

where

$$d_1 = \frac{\ln(Se^{-iT}/Xe^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

and $d_2 = d_1 - \sigma\sqrt{T}$. A similar derivation shows that the value of a European-style contingent pay put option is

$$p_{BSM} = \frac{p}{e^{-rT}N(-d_2)} = \frac{Xe^{-rT}N(-d_2) - Se^{-iT}N(-d_1)}{e^{-rT}N(-d_2)} \tag{8.12}$$

ILLUSTRATION 8.4 Value contingent pay put option.

Compute the value of a three-month European-style contingent pay stock index put option with an exercise price of 50. Assume the index has a level of 49, a dividend yield of 2%, and a volatility rate of 20%. The risk-free rate is 5%.

First, the values of the prepaid forward and exercise prices in the contingent pay put option are

$$Se^{-iT} = 49e^{-0.02(0.25)} = 48.756$$
$$Xe^{-rT} = 50e^{-0.05(0.25)} = 49.379$$

compute the values of d_1 and d_2,

$$d_1 = \frac{\ln(48.756/49.379) + 0.5(0.20)^2(0.25)}{0.20\sqrt{0.25}} = -0.0770$$

$$d_2 = -0.0770 - 0.20\sqrt{0.25} = -0.1770$$

and identify the cumulative normal probabilities,

$$N(0.0770) = 0.5307 \quad \text{and} \quad N(0.1770) = 0.5703$$

Gather the terms and compute the standard European-style put option value, that is,

$$p = 49.379(0.5703) - 48.756(0.5307) = 2.284$$

The value of the contingent pay put option is

$$p_{\text{contingent pay}} = \frac{2.284}{e^{-0.05(0.25)}(0.5703)} = 4.056$$

The values of contingent pay calls and puts can be solved using the OPTVAL function,

OV_NS_CONTINGENT_PAY_OPTION(*s, x, t, r, i, v, cp*)

where all function parameter notation is as defined earlier. Thus

OV_NS_CONTINGENT_PAY_OPTION(49, 50, 0.25, 0.05, 0.02, 0.25, "c") = 3.913

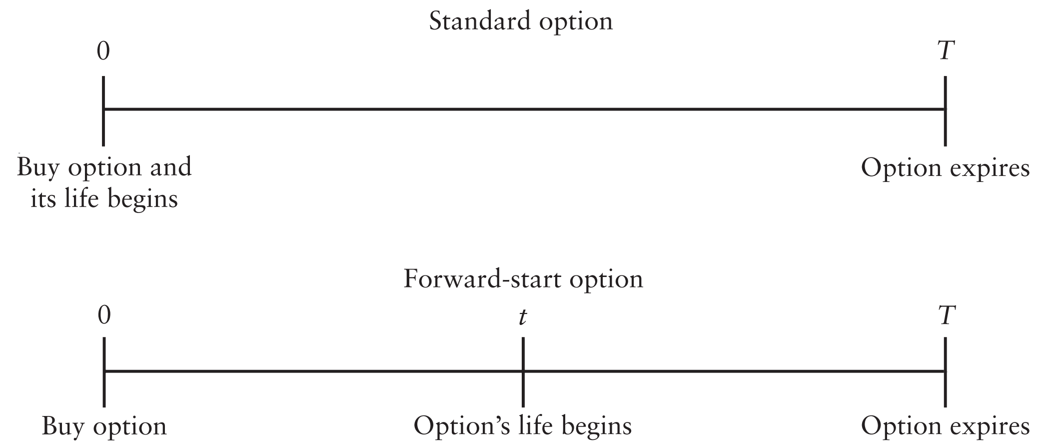
and

OV_NS_CONTINGENT_PAY_OPTION(49, 50, 0.25, 0.05, 0.02, 0.25, "p") = 4.056

FORWARD-START OPTIONS

A *forward-start option* is like a standard option with exercise price X and time to expiration T, except that the option's life begins only after prespecified period t. Figure 8.8 illustrates. Buying a standard option means paying for the option today and having its life begin. Buying a forward-start option means paying for the option today but having its life begin at time t. Thus, at time 0, the forward-start option's time to expiration is T, and, at time t, it is $T - t$. Another distinction is that a forward-start option's exercise price is set at time t. By convention, the exercise price is set equal to some positive constant times α the prevailing

FIGURE 8.8 Comparison of standard and forward-start option lives.



asset price at time t. Where $\alpha = 1$, the option will be at-the-money at time t. Where $\alpha > 1$, the call will be out of the money and the put will be in the money, and, where $\alpha < 1$, the call will be in the money and the put will be out of the money.

To value a forward-start option, we must account for the fact that the underlying asset price continues to appreciate over the time between now and when the option's life begins.[6] In a risk-neutral world, the expected asset price at time t equals the forward price of the asset, that is, $E(\tilde{S}_t) = F = Se^{(r-i)t}$. The expected exercise price of the forward-start option is therefore $\alpha E(\tilde{S}_t) = \alpha Se^{(r-i)t}$. To value a forward-start call option, we replace the asset price S and exercise price X in the BSM call option formula (8.3) with the prepaid forward price Se^{-it} and prepaid exercise price αSe^{-it}. The value of a European-style forward-start call option is

$$c_{\text{forward-start}} = Se^{-it}e^{-i(T-t)}N(d_1) - \alpha Se^{-it}e^{-i(T-t)}N(d_2) \tag{8.13}$$

where

$$d_1 = \frac{\ln(e^{-iT}/\alpha e^{-rT}) + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

and $d_2 = d_1 - \sigma\sqrt{T-t}$. The value of a forward-start put option is

$$p_{\text{forward-start}} = Se^{-it}[\alpha e^{-r(T-t)}N(-d_2) - e^{-i(T-t)}N(-d_1)] \tag{8.14}$$

[6] Valuation equations for forward-start options are developed in Rubinstein (1991a).

ILLUSTRATION 8.5 Value forward-start call option.

Compute the value of a nine-month European-style call option that is 10% out-of-the-money and compare it to the value of a nine-month European-style call option that begins in three months. Assume the underlying asset has a price of 60, a dividend yield of 1%, and a volatility rate of 30%. The risk-free rate is 4%.

The value of the forward-start call option may be computed as follows:

$$c_{\text{forward-start}} = 60e^{-0.01(0.25)}[e^{-0.01(0.5)}N(d_1) - 1.1e^{-0.04(0.5)}N(d_2)] = 3.120$$

where

$$d_1 = \frac{\ln(60e^{-0.01(0.5)}/60(1.1)e^{-0.04(0.5)}) + 0.5(0.30^2)0.5}{0.30\sqrt{0.5}} = -0.2725$$

$d_2 = -0.2725 - 0.30\sqrt{0.5} = -0.4847$, $N(d_1) = 0.3926$, and $N(d_2) = 0.3140$.
Its value can be confirmed using the OPTVAL function

OV_NS_FORWARD_START_OPTION(*s, alpha, td, t, r, i, v, cp*)

where *alpha* is a positive constant that sets the exercise price of the option relative to the asset price at the beginning of the forward-start period, *td* is the time until the beginning of the forward-start period, and all other function notation is as defined earlier. For the forward-start call in this illustration,

OV_ NS_FORWARD_START_OPTION (60, 1.1, 0.25, 0.75, 0.04, 0.01, 0.30, “c”)
= 3.120

The value of an ordinary European-style call option with nine months to expiration is

$$c = 60[e^{-0.01(0.75)}N(d_1) - 1.1e^{-0.04(0.75)}N(d_2)] = 4.386$$

where

$$d_1 = \frac{\ln(e^{-0.01(0.75)}/1.1e^{-0.04(0.75)}) + 0.5(0.30^2)0.75}{0.30\sqrt{0.75}} = -0.1503$$

$$d_2 = -0.1503 - 0.30\sqrt{0.75} = -0.4102$$

$N(d_1) = 0.4402$, and $N(d_2) = 0.3408$. The forward-start European-style call has lower value because, although the underlying asset price is expected to be the same at the end of nine months, the range of possible option prices in nine months is smaller for the forward start call than the standard call.

RATCHET OPTIONS

A *ratchet option* (also called a *cliquet option*) is a sequence of forward-start options. At the end of each option’s life a new option is written at a strike price equal to the prevailing asset times the preset constant, α. A one-year ratchet option with monthly payments will normally have 12 payments (exercise dates) equal to the maximum of the asset price less the exercise price or zero. The exercise price of each one month option is usually set at the beginning of each period. Thus the exercise price of the first option is known today and equals αS. The overall value of the ratchet call option is the sum of the values of its forward-start call options, that is,

$$c_{\text{ratchet}} = \sum_{i=1}^{n} Se^{-it_i}[e^{-i(T_i - t_i)}N(d_{1,i}) - \alpha e^{-r(T_i - t_i)}N(d_{2,i})] \tag{8.15}$$

where n is the number of settlements, t_i is the time to the forward start of the i-th option when the exercise price is fixed, and T_i is the time to maturity of the i-th. The upper integral limits are

$$d_{1,i} = \frac{\ln(e^{-iT_i}/\alpha e^{-rT_i}) + 0.5\sigma^2(T_i - t_i)}{\alpha\sqrt{T_i - t_i}}$$

and $d_{2,i} = d_{1,i} - \alpha\sqrt{T_i - t_i}$. The value of a ratchet put option is

$$p_{\text{ratchet}} = \sum_{i=1}^{n} Se^{-it_i}[\alpha e^{-r(T_i - t_i)}N(d_2) - e^{-i(T_i - t_i)}N(d_1)] \tag{8.16}$$

ILLUSTRATION 8.6 Value ratchet call option.

Compute the value of a 12-month European-style ratchet call option with monthly settlements. Assume the exercise price of each option is set at the beginning of the month. Assume the call is written on the S&P 500 index. The level of the index is 1,150, its dividend yield rate is 1%, and its volatility rate is 20%. The risk-free rate is 4%.

The value of the first forward-start call in the series is

$$c_1 = 1{,}150e^{-0.01(1/12)}[e^{-0.01(1/12)}N(d_1) - e^{-0.04(1/12)}N(d_2)] = 27.888$$

where

$$d_1 = \frac{\ln(e^{-0.001(1/12)}/e^{-0.04(1/12)}) + 0.5(0.20^2)(1/12)}{0.20\sqrt{1/12}} = 0.0722$$

$d_2 = 0.0722 - 0.20\sqrt{1/12} = 0.0144$, $N(d_1) = 0.5288$, and $N(d_2) = 0.5058$. This can be confirmed using the OPTVAL function,

OV_NS_FORWARD_START_OPTION(1150, 1, 0, 1/12, 0.04, 0.01, 0.20, “c”)
= 27.888

The value of the second forward-start call in the series is

$$c_2 = 1{,}150e^{-0.01(1/12)}[e^{-0.01(1/12)}N(d_1) - e^{-0.04(1/12)}N(d_2)] = 27.865$$

or

OV_NS_FORWARD_START_OPTION(1150, 1, 1/12, 2/12, 0.04, 0.01, 0.20, “c”)
= 27.865

and so on. The value of the ratchet option in total can easily be computed in using Excel and the forward-start option valuation function as is illustrated in the table below.

Month	t_i	T_i	Forward Start Option
1	0.00000	0.08333	27.888
2	0.08333	0.16667	27.865
3	0.16667	0.25000	27.842
4	0.25000	0.33333	27.819
5	0.33333	0.41667	27.796
6	0.41667	0.50000	27.772
7	0.50000	0.58333	27.749
8	0.58333	0.66667	27.726
9	0.66667	0.75000	27.703
10	0.75000	0.83333	27.680
11	0.83333	0.91667	27.657
12	0.91667	1.00000	27.634
Value of ratchet option			333.132

The value of the ratchet option can also be computed using

OV_NS_RATCHET_OPTION(*s*, *alpha*, *td*, *tb*, *n*, *r*, *i*, *v*, *cp*),

where *s* is the current asset price, *alpha* is the exercise price expressed as a proportion of the prevailing asset price, *td* is the time until the first option expires, *tb* is the time between reset dates, *n* is the number of reset dates, *r* is the risk-free rate of interest, *i* is the income rate on the asset, *v* is the asset’s volatility rates, and *cp* is a call/put indicator variable. Thus

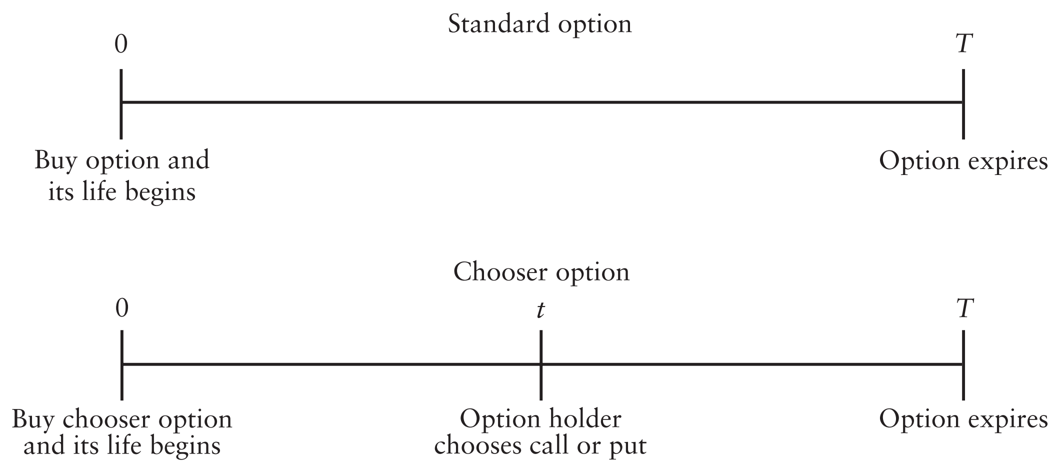
OV_NS_RATCHET_OPTION(1150, 1, 1/12, 1/12, 12, 0.04, 0.01, ,20, “c”) = 333.132

CHOOSER OPTIONS

A *chooser option* is an option that gives its holder the right to choose whether the option is to be a standard call or put after time *t*, where the call and the put have the same exercise price *X* and time to maturity *T*.[7] Figure 8.9 compares the chooser option’s life with that of a standard option. In buying a standard option, the option buyer makes an irrevocable decision to buy a call or a put. In buying a chooser option, the option buyer is allowed the additional privilege of being able to decide between the call and the put at prespecified date during the option’s life.

[7] The valuation equation of a European-style chooser option first appeared in Rubinstein (1991b). Rubinstein also values a *complex chooser option* that provides its holder with the choice between a call and a put at time *t*, however, the call and put have different exercise prices and time to expiration.

FIGURE 8.9 Comparison of standard and chooser option lives.



To value the chooser option, it is best to first focus on the value of the chooser option at time t, that is,

$$\max[c_{BSM}(S_t, X, T-t), p_{BSM}(S_t, X, T-t)] \tag{8.17}$$

where $c_{BSM}(S,X,T-t)$ and $p_{BSM}(S_t,X,T-t)$ are the BSM call and put valuation equations evaluated at time t with uncertain asset price S_t. Note that, by virtue of put-call parity, expression (8.17) may be rewritten as

$$\begin{aligned} &\max[c_{BSM}(S_t, X, T-t), (c_{BSM}(S_t, X, T-t) - S_t e^{-i(T-t)} + Xe^{-r(T-t)})] \\ &= c_{BSM}(S_t, X, T-t) + \max[0, (Xe^{-r(T-t)} - S_t e^{-i(T-t)})] \end{aligned} \tag{8.18}$$

To value a chooser option, consider the value at time t of a replicating portfolio that involves buying a standard European-style call with exercise price X and time to expiration T, $c_{BSM}(S,X,T-t)$, and a standard European-style put option with an exercise price of $Xe^{-r(T-t)}$ and a time to expiration of t whose underlying asset price is $Se^{-i(T-t)}$, $p_{BSM}(Se^{-i(T-t)},Xe^{-r(T-t)},t)$. At time t, the call option has a value of $c_{BSM}(S,X,T-t)$, and the put option has a value of (a) 0 if $Se^{-i(T-t)} \geq Xe^{-r(T-t)}$, and (b) $Xe^{-r(T-t)} - Se^{-i(T-t)}$ if $Se^{-i(T-t)} < Xe^{-r(T-t)}$. Thus, we have mimicked the payoffs in (8.18). The value of a European-style chooser option is therefore

$$\begin{aligned} c_{\text{chooser}}(S, X, t, T) = {} & Se^{-iT}N(d_1) - Xe^{-rT}N(d_2) \\ & - Se^{-iT}N(-d') + Xe^{-rT}N(-d'_2) \end{aligned} \tag{8.19}$$

where

$$d_1 = \frac{\ln(Se^{-iT}/Xe^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}, \; d_2 = d_1 - \sigma\sqrt{T-t}$$

$$d_1' = \frac{\ln(Se^{-iT}/Xe^{-rT}) + 0.5\sigma^2 t}{\sigma\sqrt{t}},\ d_2' = d_1' - \sigma\sqrt{T-t}$$

Note that, where $t = T$, the chooser has the same value as a European-style straddle (i.e., equals the sum of the values of a standard European-style call and put with exercise price X and time to expiration T). Note also that as $t \to 0$, the values of $N(-d_1')$ and $N(-d_2')$ approach 0 or 1, depending upon whether Se^{-iT} is greater than or less than Xe^{-rT}. If $Se^{-iT} > Xe^{-rT}$, $N(-d_1')$ and $N(-d_2')$ are 0, in which case the last two terms in (8.19) disappear and the lower price bound of the chooser is the standard European-style call option value. If $Se^{-iT} < Xe^{-rT}$, $N(-d_1')$ and $N(-d_2')$ are 1, in which case the last two terms in (8.19) become $-Se^{-iT} + Xe^{-rT}$, and the lower price bound of the chooser is the standard European-style put option value, that is,

$$\begin{aligned} c_{\text{chooser}}(S, X, t, T) &\geq Se^{-iT}N(d_1) - Xe^{-rT}N(d_2) - Se^{-iT} + Xe^{-rT} \\ &= Xe^{-rT}N(-d_2) - Se^{-iT}N(-d_1) \end{aligned}$$

ILLUSTRATION 8.7 Value chooser option.

Compute the value of a one-year European-style chooser option that allows you to choose whether the option is a call or a put at the end of three months. Assume the option is written on the S&P 500 index portfolio, and that the S&P 500 index has a current level of 1,100, a dividend yield rate of 1%, and a volatility rate of 15%. Assume that the exercise price of the chooser is 1,150 and that risk-free rate of interest is 4%.

First, compute the prepaid forward price of the stock index and the prepaid exercise price.

$$Se^{-iT} = 1{,}100e^{-0.01(1)} = 1{,}089.05$$

and

$$Xe^{-rT} = 1{,}150e^{-0.04(1)} = 1{,}104.91$$

Second, compute the upper integral limits.

$$d_1 = \frac{\ln(1{,}089.05/1{,}104.91) + 0.5(0.15^2)(1)}{0.15\sqrt{1}} = -0.0213$$

$$d_2 = -0.0213 - 0.15\sqrt{1} = -0.1713$$

$$d_1' = \frac{\ln(1{,}089.05/1{,}104.91) + 0.5(0.15^2)(0.25)}{0.15\sqrt{0.25}} = -0.1552$$

$$d_2' = -0.1552 - 0.15\sqrt{0.25} = -0.2302$$

Third, compute the respective risk-neutral probabilities.

$$N(d_1) = 0.4915,\ N(d_2) = 0.4320,\ N(-d_1) = 0.5617, \text{ and } N(-d_2) = 0.5910$$

Finally, compute the chooser option value.

$$\begin{aligned} c_{\text{chooser}} &= 1{,}089.05(0.4915) - 1{,}104.91(0.4320) - 1{,}089.05(0.5617) + 1{,}104.91(0.5910) \\ &= 99.3086 \end{aligned}$$

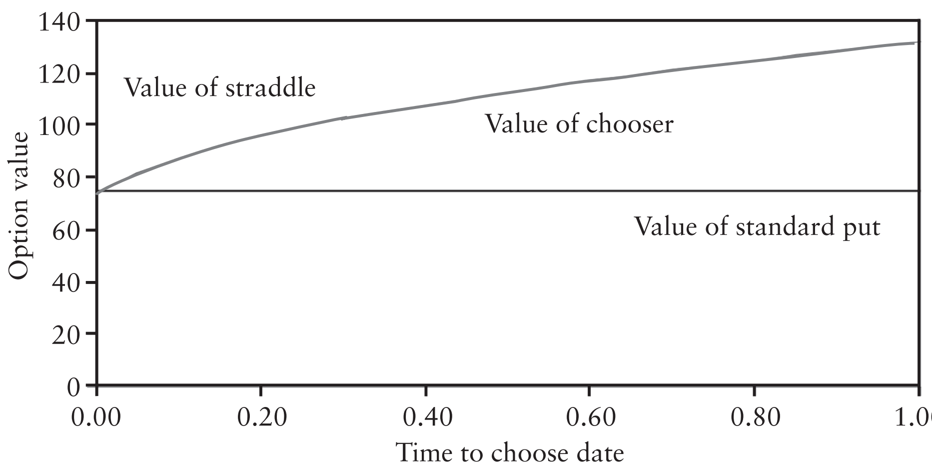
To verify the computation, use the OPTVAL function

OV_NS_CHOOSER_OPTION(*s, x, td, t, r, i, v*)

where *td* is the time until the choice between the call and the put must be made, and all other notation has been defined. Thus

OV_NS_CHOOSER_OPTION(1100, 1150, 0.25, 1., 0.04, 0.01, 0.15) = 99.3086

Note that the values of standard call and put options with one year to expiration are 57.9604 and 73.8134, respectively. The chooser option is more valuable since it allows the holder to choose between the call and put after three months of asset price movement has elapsed rather than now. Since the put has the highest value of the standard options today, it serves as the lower value bound of the chooser option. When the time until the choose date is 0, $c_{\text{chooser}} = 73.8134$. When the time until the choose date equals one year, the chooser value equals the value of a straddle, that is, the sum of the standard call and put values, 131.7737. The value of the chooser option rises at a decreasing rate as the time until the choose date approached the option's time to expiration, as is shown in the following figure.



EXCHANGE OPTIONS

An *exchange option*[8] is the right to exchange one asset for another. The value of the right to exchange asset 2 for asset 1 (i.e., a call option to buy conveying the right to buy asset 1 by paying asset 2 is

[8] The exchange option formula for the two-asset case where both assets have a cost of carry rate equal to the risk-free rate of interest was derived by Margrabe (1978). The formula presented here generalizes the Margrabe result to allow the assets to have different income rates. The *n*-asset exchange option was later developed by Margrabe (1982).

$$c_{\text{exchange}}(S_1, S_2) = S_1 e^{-i_1 T} N_1(d_1) - S_2 e^{-i_2 T} N_1(d_2) \tag{8.20}$$

where

$$d_1 = \frac{\ln(S_1 e^{-i_1 T} / S_2 e^{-i_2 T}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

S_1 and S_2 are the underlying asset prices, T is the time between now and the expiration date, i_1 and i_2 are the income rates (e.g., dividend yields) of asset 1 and asset 2, σ_1 and σ_2 are the expected future volatility rates of assets 1 and 2, and ρ is the expected correlation between the returns of assets 1 and 2. The term $N(d_2)$ is the risk-neutral probability that the price of asset 1 will exceed the price of asset 2 at expiration. As noted in Chapter 3, the terms,

$$S_1 e^{-i_1 T} \text{ and } S_2 e^{-i_2 T}$$

are the prices of prepaid forward contracts on assets 1 and 2, respectively.

The exchange option formula (8.20) is interesting in a number of respects. First, it contains the BSM call option formula as a special case. In the BSM model, the prepaid forward price on the asset 2 is the present value of the exercise price, that is,

$$Se^{-i_2 T} = Xe^{-rT}$$

The expression for d_1 can be rearranged to yield the more familiar

$$d_1 = \frac{\ln(S_1 e^{-i_1 T} / Xe^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

And, since asset 2 is risk-free, the volatility rate becomes

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \sigma_1$$

Another interesting aspect of (8.20) is that the value of a call option to "buy" asset 1 with asset 2, $c_{\text{exchange}}(S_1,S_2)$, equals the value of a put option to "sell" asset 2 for asset 1, $p_{\text{exchange}}(S_2,S_1)$. In the case of the call, the option is exercised at expiration if the price of asset 1 exceeds the price of asset 2; otherwise, it expires worthless. In the case of the put, the option is exercised at expiration if the price of asset 2 is less than the price of asset 1; otherwise it expires worthless. Since the payoffs of these two options are identical, the value of the options must be the same, that is,

$$c_{\text{exchange}}(S_1,S_2) = p_{\text{exchange}}(S_2,S_1)$$

The value of a put option to "sell" asset 1 for asset 2, on the other hand, is

$$p_{\text{exchange}}(S_2, S_1) = S_2 e^{-i_2 T} N_1(-d_2) - S_1 e^{-i_1 T} N_1(-d_1) \tag{8.21}$$

where d_1 and d_2 are as defined below equation (8.20). It is equivalent in value to a call option to exchange asset 1 for asset 2.

Exchange options are traded primarily in the OTC market. The underlying assets can be virtually any financial or commodity. Exchange options are also often embedded in other financial contracts. The CBT's corn futures contract, for example, calls for the delivery of No. 2 yellow corn at par, but also permits the delivery of No. 3 yellow at a 1½ cent discount below the contract price. Thus, standing prior to the delivery day, an individual who is short the corn futures contract and holds No. 2 yellow has the right to deliver No. 3 yellow and will do so if the price difference between No. 2 yellow and No. 3 yellow is greater than 1½ cents. We will discuss this particular application in Chapter 20.

ILLUSTRATION 8.8 Value exchange call option.

Compute the value of a three-year European-style exchange call option that allows you to exchange one unit of the DJIA index level for ten units of the S&P 500 index level. Assume the S&P 500 portfolio has a current level of 1,150, a dividend yield rate of 1%, and a volatility rate of 20%. Assume the DJIA has a current level of 10,500, a dividend yield rate of 2%, and a volatility rate of 18%. Finally, assume the correlation between the returns of the two indexes is .85. Also, compute the value of a three-year European-style put option that allows you to sell 10 units of the S&P 500 index and receive one unit of the DJIA. Comment on the difference between the two option values.

The values of the prepaid forwards that appear in the exchange option formula are

$$S_1 e^{-i_1 T} = 10 \times 1{,}150 \times e^{-0.01(3)} = 11{,}160.12$$

and

$$S_2 e^{-i_2 T} = 10{,}500 \times e^{-0.02(3)} = 9{,}888.53$$

Since the first prepaid forward has a higher value than the second, the call is currently in the money. The value of the exchange call option is

$$c_{\text{exchange}}(S_1, S_2) = 11{,}160.12 N_1(d_1) - 9{,}888.53 N_1(d_2) = 1{,}565.19$$

where

$$d_1 = \frac{\ln(11{,}160.12/9{,}888.53) + 0.5\sigma^2(3)}{\sigma\sqrt{3}}, \; d_2 = d_1 - \sigma\sqrt{3}$$

$$\sigma = \sqrt{0.20^2 + 0.18^2 - 2(0.85)(0.20)(0.18)} = 0.106$$

Its value may be computed using the OPTVAL function

OV_NS_EXCHANGE_OPTION(*s1*, *s2*, *t*, *i1*, *i2*, *v1*, *v2*, *rho*, *cp*)

where *s1* and *s2* are the prices of assets 1 and 2, *t* is the time to expiration, *i1* and *i2* are the income rates of assets 1 and 2, *v1* and *v2* are the volatility rates of assets 1 and 2, rho is the correlation between the returns of assets 1 and 2, and *cp* is a (c)all/(p)ut indicator. Thus

OV_NS_EXCHANGE_OPTION(11500, 10500, 0.01, 0.02, 0.20, 0.18, 0.85, "c")
= 1,565.19

The value of the corresponding European-style exchange put option can be computed in the same manner:

OV_NS_EXCHANGE_OPTION(11500, 10500, 0.01, 0.02, 0.20, 0.18, 0.85, "p")
= 293.59

The difference between the prices is 1,565.19 – 293.59 = 1,271.60. Note that this is also the difference between the two prepaid forward contract prices, 11,160.12 – 9,888.53 = 1,271.60. The reason is, of course, put-call parity. For exchange options,

$$c_{\text{exchange}}(S_2, S_1) - p_{\text{exchange}}(S_2, S_1) = S_1 e^{-i_1 T} - S_2 e^{-i_2 T}$$

OPTIONS ON THE MAXIMUM AND THE MINIMUM

Options on the maximum and minimum of two or more risky assets are closely related to exchange options.[9] In place of exchanging one asset for another, however, the option holder gets to choose between the two risky assets. A call option on the maximum of two risky assets, for example, provides its holder with the right to buy the more expensive of asset 1 and asset 2 for exercise price X at the option's expiration date, T. In this section, we provide and interpret the valuation equations for (1) a call on the maximum of two risky assets; (2) a call on the minimum of two risky assets; (3) a put on the maximum of two risky asset; and (4) a put of the minimum of two risky assets.

Call on Maximum

The payoff contingencies of a European-style call on the maximum are:

$$c_{\max, T} = \begin{cases} S_{1,T} - X, & \text{if } S_{1,T} \geq S_{2,T} \text{ and } S_{1,T} \geq X \\ S_{2,T} - X, & \text{if } S_{2,T} \geq S_{1,T} \text{ and } S_{2,T} \geq X \\ 0, & \text{if } S_{1,T} < X \text{ and } S_{2,T} < X \end{cases} \tag{8.22}$$

[9] Other names for the option on the maximum are "the better of two assets" or "outperformance options." The models presented here are on the maximum or the minimum of two risky assets, and the valuation equations are based on Stulz (1982). To generalize these models to consider three or more risky assets, see Johnson (1987).

The valuation equation for a call option on the maximum is

$$\begin{aligned} c_{\max}(S_1, S_2, X) = {} & S_1 e^{-i_1 T} N_2(d_{11}, d_1'; \rho') + S_2 e^{-i_2 T} N_2(d_{21}, d_2'; \rho_2') \\ & - X e^{-rT}[1 - N_2(-d_{12}, -d_{22}; \rho_{12})] \end{aligned} \tag{8.23}$$

where

$$d_{11} = \frac{\ln(S_1 e^{-i_1 T} / X e^{-rT}) + 0.5\sigma_1^2 T}{\sigma_1 \sqrt{T}}, \; d_{12} = d_{11} - \sigma_1 \sqrt{T}$$

$$d_{21} = \frac{\ln(S_2 e^{-i_2 T} / X e^{-rT}) + 0.5\sigma_2^2 T}{\sigma_2 \sqrt{T}}, \; d_{22} = d_{21} - \sigma_2 \sqrt{T}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$$

$$d_1' = \frac{\ln(S_1 e^{-i_1 T} / S_2 e^{-i_2 T}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}, \; d_2' = \frac{\ln(S_2 e^{-i_2 T} / S_1 e^{-i_1 T}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

$$\rho_1' = \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma}, \text{ and } \rho_2' = \frac{\sigma_2 - \rho_{12}\sigma_1}{\sigma}$$

In equation (8.23), the term, $N_2(-d_{12},-d_{22};\rho_{12})$ is the *compound* risk-neutral probability that both asset 1 and asset 2 will have prices below the exercise price at the option's expiration. It is called a compound probability because there are two sources of uncertainty. We want to know the probability that asset 1's price will be below the exercise *and* that asset 2's price will be below the exercise price. Recall $N(a)$ is the probability that a random drawing x from a univariate normal distribution have a value below a, that is, $\Pr(x \leq a) = N(a)$. Here, $N_2(a,b;\rho)$ is the probability that random drawings x and y from a bivariate normal distribution will have values below a and b, respectively, that is, $\Pr(x \leq a, y \leq b) = N_2(a,b;\rho)$, where ρ is the correlation between the random variables x and y. Where we have used the OPTVAL function OV_PROB_PRUN(*a*) to compute $N(a)$, we use OV_PROB_PRBN(*a,b,rho*) to $N_2(a,b;\rho)$. An algorithm for computing the bivariate normal probability is provided in Appendix 8.A. If $N_2(-d_{12},-d_{22};\rho_{12})$ is the compound risk-neutral probability that both asset 1 and asset 2 will have prices below the exercise price at the option's expiration, then $[1 - N_2(-d_{12},-d_{22};\rho_{12})]$ must be the risk-neutral probability that one of the two asset prices will exceed the exercise price X at time T, and $Xe^{-rT}[1 - N_2(-d_{12},-d_{22};\rho_{12})]$ is the present value of the cost of exercising the option times the risk-neutral probability that the option will be exercised.

The two remaining terms on the right-hand side of (8.23) also have economic interpretations. The term

$$S_1 e^{-i_1 T} N_2(d_{11}, d'_1; \rho'_1)$$

is the present value of the expected price of asset 1 at the option's expiration conditional upon asset 1 having a price greater than asset 2 and greater than the exercise price times the risk-neutral probability that the terminal price of asset 1 will exceed the terminal price of asset 2 and will exceed the exercise price. The term

$$S_2 e^{-i_2 T} N_2(d_{21}, d'_2; \rho'_2)$$

has a similar interpretation but for asset 2.

ILLUSTRATION 8.9 Value call on maximum.

Consider a call option that provides its holder the right to buy $100,000 worth of the S&P 500 index portfolio at an exercise price of $1,200 or $100,000 worth of a particular T-bond at an exercise price of $100, whichever is worth more at the end of three months. The S&P 500 index is currently priced at $1,080, pays dividends at a rate of 1% annually and has a return volatility of 20%. The T-bond is currently priced at $98, pays a coupon yield of 6% and has a return volatility of 15%. The correlation between the rates of return of the S&P 500 and the T-bond is 0.5. The risk-free rate of interest is 4%. Compute the value of this call option on the maximum.

Before applying the option on the maximum formula, it is important to recognize that there are two exercise prices in this problem: 1,200 for the S&P index portfolio and 100 for the T-bond. What this implies is that we can buy 100,000/1,200 = 83.333 units of the index portfolio or 100,000/100 = 1,000 units of T-bonds at the end of three months, depending on which is worth more. At this juncture, we must decide whether to work with the valuation equation (8.23) in units of the S&P 500 index portfolio, in which case we multiply the current T-bond price and its exercise price by 12 and then multiply the computed option price by 83.333, or to work with the valuation equation (8.20) in units of the T-bond, in which case we divide the current S&P 500 price and the option's S&P 500 exercise price by 12 and then multiply the computed option price by 1,000.[10]

Given the choice between methods is arbitrary and produces the same option value, we will proceed with the problem solution working in units of the S&P 500 index portfolio. We begin, therefore by adjusting the T-bond prices. The current T-bond price is assumed to be 1,176 and the T-bond exercise price is 1,200. With the units of the two underlying assets comparable, we now compute the prepaid forward prices of assets 1 and 2 as well as the prepaid exercise price, that is,

$$S_1 e^{-i_1 T} = 1{,}080 e^{-0.01(0.25)} = 1{,}077.30$$
$$S_2 e^{-i_2 T} = 1{,}176 e^{-0.06(0.25)} = 1{,}158.49$$

and

$$X e^{-rT} = 1{,}200 e^{-0.04(0.25)} = 1{,}188.06$$

[10] These types of adjustments can be made freely because the option price is linearly homogeneous in both the asset price and the exercise price. See Merton (1973).

Substituting the problem parameters into equation (8.23), we get

$$\begin{aligned} c_{\max}(S_1, S_2, X) &= 1{,}077.30 \times N_2(d_{11}, d_1'; \rho_1') + 1{,}158.49 \times N_2(d_{21}, d_2'; \rho_2') \\ &\quad - 1{,}188.06 \times [1 - N_2(-d_{12}, -d_{22}'; \rho_{12}')] \\ &= 27.239 \end{aligned}$$

where

$$d_{11} = \frac{\ln(1{,}077.30/1{,}188.06) + 0.5(0.20^2)(0.25)}{0.20\sqrt{0.25}} = -0.9296$$

$$d_{12} = -0.9296 - 0.20\sqrt{0.25} = -0.2985$$

$$d_{21} = \frac{\ln(1{,}058.49/1{,}188.06) + 0.5(0.15^2)(0.25)}{0.15\sqrt{0.25}} = -1.0286$$

$$d_{22} = -1.0286 - 0.15\sqrt{0.25} = -0.3735$$

$$d_1' = \frac{\ln(1{,}077.30/1{,}058.49) + 0.5\sigma^2(0.25)}{\sigma\sqrt{0.25}} = -0.7610$$

$$d_2' = -(-0.7610 - \sigma\sqrt{0.25}) = 0.8511$$

$$\sigma = \sqrt{0.20^2 + 0.15^2 - 2(0.5)(0.20)(0.15)} = 0.1803$$

$$\rho_1' = \frac{0.20 - 0.5(0.15)}{0.1803} = 0.6934$$

and

$$\rho_2' = \frac{0.15 - 0.5(0.20)}{0.1803} = 0.2774$$

The risk-neutral probabilities are $N_2(d_{11}, d_1'; \rho_1') = 0.1102$, $N_2(d_{21}, d_2'; \rho_2') = 0.3355$, and $N_2(-d_{12}, -d_{22}; \rho_{12}) = 0.5958$. The computed option value is 27.239. This can be confirmed using the OPTVAL function

OV_NS_MAXMIN_OPTION(*s1, s2, x, t, r, i1, i2, v1, v2, rho, cp, mm*),

where *s1* and *s2* are the prices of assets 1 and 2, *x* is the exercise price of the option, *t* is the time to expiration, *i1* and *i2* are the income rates of assets 1 and 2, *v1* and *v2* are the volatility rates of assets 1 and 2, *rho* is the correlation between the returns of assets 1 and 2, *cp* is a (c)all/(p)ut indicator, and *mm* is a ma(x)imum/mi(n)imum indicator. Thus

OV_NS_MAXMIN_OPTION(1080, 1176, 1200, 0.25, 0.04, 0.01, 0.06, 0.20, 0.15, 0.5, “c”, “x”) = 27.239

which implies the total value of the option contract is 27.239 × 83.333 = 2,269.90. The probability that either or both the components of the option are in-the-money at expiration is $[1 - N_2(1.0286, 0.3735; 0.5)] = 0.4042$ or 40.42%.

Before turning to the call on the minimum, it is worthwhile to note that the formula for the call on the maximum becomes the exchange option formula when the exercise price of the option is zero. Where $X = 0$, equation (8.23) may be written

$$\begin{aligned} c_{\max}(S_1, S_2, 0) &= S_1 e^{-i_1 T} N_2(\infty, d_1'; \rho_1') + S_2 e^{-i_2 T} N_2(\infty, d_2'; \rho_2') \\ &\quad - 0e^{-rT}[1 - N_2(-\infty, -\infty; \rho_{12})] \\ &= S_1 e^{-i_1 T} N_1(d_1') + S_2 e^{-i_2 T} N_1(d_2') \end{aligned}$$

which is the exchange option formula (8.20) presented earlier.

Call on Minimum

The payoff contingencies of a European-style call on the minimum are:

$$c_{\min, T} = \begin{cases} S_{1,T} - X, & \text{if } S_{1,T} \le S_{2,T} \text{ and } S_{1,T} \ge X \\ S_{2,T} - X, & \text{if } S_{2,T} \le S_{1,T} \text{ and } S_{2,T} \ge X \\ 0, & \text{if } S_{1,T} < X \text{ or } S_{2,T} < X \end{cases} \tag{8.24}$$

Under the BSM assumptions, the value of a European-style call on the minimum is

$$\begin{aligned} c_{\max}(S_1, S_2, X) &= S_1 e^{-i_1 T} N_2(d_{11}, -d_1'; -\rho_1') + S_2 e^{-i_2 T} N_2(d_{21}, -d_2'; -\rho_2') \\ &\quad - Xe^{-rT} N_2(d_{12}, d_{22}; -\rho_{12}) \end{aligned} \tag{8.25}$$

where all notation is as previously defined. Note that, unlike the call on the maximum of two risky assets, the risk-neutral probability in (8.25), $N_2(d_{12}, d_{22}; \rho_{12})$, requires that both asset prices exceed the exercise price X at time T. If one of the terminal asset prices is below X at time T, the call on the minimum expires worthless. It is also worthwhile to note that the sum of the payoffs of a call on the maximum and a call on the minimum are

$$\begin{aligned} c_{\max, T} + c_{\min, T} &= \max[\max(S_{1,T}, S_{2,T}) - X, 0] + \max[\min(S_{1,T}, S_{2,T}) - X, 0] \\ &= \max[S_{1,T} - X, 0] + \max[S_{2,T} - X, 0] \end{aligned}$$

In absence of costless arbitrage opportunities, this means that the sum of the values a call on the maximum and a call on the minimum is equal to the sum of the values of standard call options written on the individual assets, that is,

$$c_{\max}(S_1, S_2, X) + c_{\min}(S_1, S_2, X) = c_{BSM}(S_1, X) + c_{BSM}(S_2, X) \tag{8.26}$$

ILLUSTRATION 8.10 Value call options on maximum and minimum.

Consider call options on the minimum and the maximum of one share of ABC and DEF shares. The options' exercise prices are 50, and their time to expiration is six months. ABC is currently priced at 51, pays dividends at a rate of 1% annually, and has a return *volatility of 35%. DEF is currently priced at 49, pays a dividend yield of 3% and has a return volatility of 32%. The correlation between the rates of return of ABC and DEF is 0.75. The risk-free rate of interest is 4%. Compute the value of the call option on the minimum. Also, compute the value of a call option on the maximum, and the values of standard call options on the individual shares.*

Given that we showed all the underpinnings of an option on the minimum/maximum in Illustration 8.9, we will simply apply the appropriate OPTVAL functions to value the options. The call option on the minimum has a value,

OV_NS_MAXMIN_OPTION(51, 49, 50, 0.5, 0.04, 0.01, 0.03, 0.35, 0.32, .5, "c", "n")
= 2.932,

and the call option on the maximum is

OV_NS_MAXMIN_OPTION(51, 49, 50, 0.5, 0.04, 0.01, 0.03, 0.35, 0.32, 0.5, "c", "n")
= 6.917.

The sum of the values is 2.932 + 6.917 = 9.850. The values of standard call options written on the shares of ABC and DEF are

OV_OPTION_VALUE (51, 50, 0.5, 0.04, 0.01, 0.35, "c", "e") = 5.828

and

OV_OPTION_VALUE (49, 50, 0.5, 0.04, 0.03, 0.32, "c", "e") = 4.022

The sum of the values of the standard call options is also 9.850, verifying the no-arbitrage condition (8.26).

Put on Maximum

The value of a European-style put on the maximum is

$$
\begin{aligned}
p_{\max}(S_1, S_2, X) = {} & Xe^{-rT}N_2(-d_{12}, -d_{22}; \rho_{12}) - S_1 e^{-i_1 T} N_2(-d_{11}, -d_1'; -\rho_1') \\
& - S_2 e^{-i_2 T} N_2(-d_{21}, d_2'; -\rho')
\end{aligned}
\tag{8.27}
$$

Similar to the situation with the call on the minimum, $N_2(d_{12},d_{22};\rho_{12})$ is the risk-neutral probability that both asset prices are below the exercise price X at time T. If one of the terminal asset prices is above X at time T, the put on the maximum expires worthless.

Put on Minimum

The value of a European-style put on the minimum is

$$
\begin{aligned}
p_{\min}(S_1, S_2, X) = {} & Xe^{-rT}N_2(-d_{12}, -d_{22}; \rho_{12}) - S_1 e^{-i_1 T} N_2(-d_{11}, -d_1'; -\rho_1') \\
& - S_2 e^{-i_2 T} N_2(-d_{21}, d_2'; -\rho')
\end{aligned}
\tag{8.28}
$$

where all notation is as previously defined. In (8.28), the term, $[1 - N_2(d_{12},d_{22};\rho_{12})]$, is the risk-neutral probability that one of the two asset prices will be below the exercise price X at time T or, alternatively, one minus the probability that both asset prices will exceed the exercise price at the option's expiration. Like in the case of the call, the sum of the values of a put on the maximum and a put on the minimum equals the sum of the values of standard put options written on the individual assets, that is,

$$p_{\max}(S_1, S_2, X) + p_{\min}(S_1, S_2, X) = p_{BSM}(S_1, X) + p_{BSM}(S_2, X) \quad (8.29)$$

COMPOUND OPTIONS

A *compound option* is an option on an option. It is like a standard option in the sense that it conveys the right to buy or sell an underlying asset at the contract's expiration. The only difference is that the underlying asset happens to be an option. Compound options are traded in the OTC market. The most common forms include calls on calls, puts on calls, calls on puts, and puts on puts. We will address each in turn.[11]

Call on Call

A call on a call conveys the right to buy an underlying call option with exercise price X and time to expiration T. The call on the call (i.e., the compound option) has exercise price c^* and time to expiration t. Its value is denoted c_{call}. Under risk-neutral valuation, a call on a call may be written

$$c_{\text{call}}(c^*, t) = e^{-rt}E(\tilde{c}_t) \quad (8.30)$$

where c_t is the value of the underlying call at time t,

$$c_{\text{call}}(c^*, t) = \begin{cases} c(S_t, T-t, X) & \text{if } c_t > c^* \\ 0 & \text{if } c_t \le c^* \end{cases}$$

$$c(S_t, T-t, X) = S_t e^{-i(T-t)} N_1(d_1) - X e^{-r(T-t)} N_1(d_2) \quad (8.31)$$

$$d_1 = \frac{\ln(S_t e^{-i(T-t)}/X e^{-r(T-t)}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}, \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

[11] The formulas in this section are based on Geske (1979). Interestingly, he derived the formulas under the BSM assumptions before the instruments ever traded in the OTC market. His application arose from the observation that the equity of a firm can be viewed as a call option on the value of the firm with the exercise being equal to the value of the firm's bonds. Consequently, exchange-traded call and put options on the shares of the firm are actually a call on a call and a put on a call.

The first step in valuing a call on a call is determining the critical asset price at time t above which you will exercise the compound option at time t. It can be determined by iteratively searching for the asset price S_t^* that makes the value of the underlying call equal to the exercise price of the compound call, that is,

$$c(S_t^*, T-t, X) = c^* \tag{8.32}$$

With S_t^* known, the value of a European-style call on a call is

$$c_{\text{call}}(c^*, t) = Se^{-iT}N_2(a_1, b_1; \rho) - Xe^{-rT}N_2(a_2, b_2; \rho) - e^{-rt}c^*N_1(b_2) \tag{8.33}$$

where

$$a_1 = \frac{\ln(Se^{-iT}/Xe^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}},\ a_2 = a_1 - \sigma\sqrt{T}$$

$$b_1 = \frac{\ln(Se^{-iT}/S_t^*e^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}},\ b_2 = b_1 - \sigma\sqrt{T},\ \rho = \sqrt{\frac{t}{T}}$$

and $N_1(.)$ and $N_2(.)$ are the cumulative univariate and bivariate unit normal density functions.

Notice the similarity between the structure of (8.33) and the structure of the BSM call option formula (8.3). The first two terms of the right-hand side of (8.33) correspond to the BSM formula. Instead of getting the underlying asset upon exercising a standard call, you get a call option. The last term on the right-side is the present value of the exercise price of the compound option, $e^{-rt}c^*$, times the risk-neutral probability that the asset price will exceed the critical asset price at time t, $N_1(b_2)$, or the expected cost of exercising the compound call conditional upon it being in the money at time t. The term, $N_2(a_2,b_2;\rho)$, is the risk-neutral compound probability that the asset price will exceed S_t^* at time t and will exceed the exercise price X at time T. The asset price must jump both hurdles to be in-the-money at time T. The sign of correlation coefficient, ρ, reflects whether asset price should move in the same or opposite direction in the interval between time 0 and time t as in the interval between time t and time T in order for the underlying option to be in-the-money at time T. For a call on a call, the sign is positive because you want the asset price to increase in both intervals. For a put on a call, the sign will be negative because you want the asset price to be low enough for the compound option to be exercised at time t and yet be high enough to exceed the exercise price of the underlying call at time T.

Put on Call

A put on a call conveys the right to sell an underlying call option with exercise price X and time to expiration T. The put has exercise price c^* and time to expiration t. The simplest way to derive the valuation equation for a put on a call is to begin with a compound option version of put-call parity. From Chapter 5, we know that, for a nonincome producing asset,[12] the call price less the put price equals the asset price less the present value of the exercise price. The equivalent condition here is that the call on a call price, $c_{call}(c^*,t)$, less the put on a call price, $p_{call}(c^*,t)$, equals the underlying European-style call option price, $Se^{-iT}N_1(a_1) - Xe^{-rT}N_1(a_2)$, less the present value of the exercise price, $e^{-rt}c^*$, that is,

$$c_{call}(c^*, t) - p_{call}(c^*, t) = Se^{-iT}N_1(a_1) - Xe^{-rT}N_1(a_2) - e^{-rt}c^* \tag{8.34}$$

To value a put on a call, we isolate the value of the put on the call and get

$$p_{call}(c^*, t) = Xe^{-rT}N_2(a_2, -b_2; -\rho) - Se^{-iT}N_2(a_1, -b_1; -\rho) + e^{-rT}c^*N_1(-b_2) \tag{8.35}$$

where all notation is defined above.[13] In (8.35), $N_1(-b_2)$ is the risk-neutral probability that the asset price will be below the critical asset price at time t, S_t^*. In this region, the compound option will be exercised. The underlying call value, however, increases with the asset price. The correlation in the compound probability then is negative, and the term, $N_2(a_2,-b_2;-\rho)$, is the risk-neutral compound probability that the asset price will be below S_t^* at time t and will exceed the exercise price X at time T.

Put on Put

A put on a put conveys the right to sell an underlying put option with exercise price X and time to expiration T. The put has exercise price of p^* and time to expiration t. Under risk-neutral valuation, the value of a put on a put is

$$p_{put}(p^*, t) = e^{-rt}E(\tilde{p}_t) \tag{8.36}$$

where p_t is the value of the underlying put at time t,

$$p_{put,\,t}(p^*, 0) = \begin{cases} p(S_t, T-t, X) & \text{if } p_t > p^* \\ 0 & \text{if } p_t \le p^* \end{cases}$$

$$p(S_t, T-t, X) = Xe^{-r(T-t)}N_1(-d_2) - S_te^{-iT}N_1(-d_1) \tag{8.37}$$

$$d_1 = \frac{\ln(S_te^{-i(T-t)}/Xe^{-r(T-t)}) + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

[12] Recall that in Chapter 5 we discussed the fact that the cost of carry rate for an option is the risk-free rate of interest.

[13] In deriving (8.32), we use some properties of the cumulative univariate and bivariate normal distributions. Specifically, $1-N_1(b) = N_1(-b)$ and $N_1(a) - N_2(a,b;-\rho)$. See Abramowitz and Stegum (1972, p. 936).

The first step in valuing a put on a put is determining the critical asset price at time t above which you will exercise your option to sell the underlying put. It can be determined by iteratively searching for the value S_t^* that satisfies

$$p(S_t^*, T-t, X) = p^* \tag{8.38}$$

With S_t^* known, the value of a European-style put on a put is

$$p_{\text{put}}(p^*, t) = Se^{-iT}N_2(-a_1, b_1; -\rho) - Xe^{-rT}N_2(-a_2, b_2; -\rho) + e^{-rt}p^*N_1(b_2) \tag{8.39}$$

where all notation is defined above. The term, $N_1(b_2)$, is the risk-neutral probability that the asset price will be above the critical asset price at time t, S_t^* (i.e., the put will be exercised), and the term, $N_2(-a_2,b_2;-\rho)$, is the risk-neutral probability that the asset price will be above S_t^* at time t and will be below the exercise price X at time T.

Call on Put

A call on a put conveys the right to buy an underlying put option with exercise price X and time to expiration T. The call has exercise price p^* and time to expiration t. Again, put-call parity can be used to arrive at the valuation formula. For a nonincome producing asset, the call price less the put price equals the asset price less the present value of the exercise price. The equivalent condition here is that the call on a put price, $c_{\text{put}}(p^*,t)$, less the put on a put price, $p_{\text{put}}(p^*,t)$, equals the underlying European-style put option price, $Xe^{-rT}N_1(-a_2) - Se^{-iT}N_1(-a_1)$, less the present value of the exercise price, $e^{-rT}p^*$, that is,

$$c_{\text{put}}(p^*, t) - p_{\text{put}}(p^*, t) = Xe^{-rT}N_1(-a_2) - Se^{-iT}N_1(-a_1) - e^{-rt}p^* \tag{8.40}$$

Rearranging to isolate the value of a European-style call on a put and simplifying,

$$c_{\text{put}}(p^*, t) = Xe^{-rT}N_2(-a_2, -b_2; \rho) - Se^{-iT}N_2(-a_1, -b_1; \rho) - e^{-rt}p^*N_1(-b_2) \tag{8.41}$$

where all other notation is as previously defined. The term, $N_1(-b_2)$, is the risk-neutral probability that the asset price will be above the critical asset price at time t, S_t^* (i.e., the put will be exercised), and the term, $N_2(-a_2,-b_2;\rho)$, is the risk-neutral probability that the asset price will be below S_t^* at time t and will be below the exercise price X at time T.

ILLUSTRATION 8.11 Value call on put.

Consider a call option that provides its holder with the right to buy a put option on the S&P 500 index portfolio. The put that underlies the call has an exercise price of 1,200 and a time to expiration of nine months. The call has an exercise price of 40 and a time to expiration of three months. The S&P 500 index is currently at 1,150, pays dividends *at a constant rate of 1% annually, and has a volatility rate of 15%. The risk-free rate of interest is 3%.*

The first step is to compute the critical asset price below which the call will be exercised at time t to take delivery of the put. This is done by solving $p(S_t^*, T-t, X) = 40.00$. The critical index level, S_t^* is 1,210.72. The next step is to apply the valuation formula. Here, we get

$$\begin{aligned} c_{\text{put}} &= 1{,}200e^{-0.03(0.75)}N_2(-a_2, -b_2; \rho) - 1{,}150e^{0.01(0.75)}N_2(-a_1, -b_1; \rho) \\ &\quad - e^{-0.03(0.25)}(40)N_1(-b_2) \\ &= 41.6110 \end{aligned}$$

where

$$a_1 = \frac{\ln(1{,}150e^{-0.01(0.75)}/1{,}200e^{-0.03(0.75)}) + 0.5(0.15^2)(0.75)}{0.15\sqrt{0.75}} = -0.1472$$

$$a_2 = -0.1472 - 0.15\sqrt{0.75} = -0.2771$$

$$b_1 = \frac{\ln(1{,}150e^{-0.01(0.25)}/1{,}210.72e^{-0.03(0.25)}) + 0.5(0.15^2)(0.25)}{0.15\sqrt{0.25}} = -0.5818$$

$$b_2 = -0.5818 - 0.15\sqrt{0.25} = -0.6568$$

$$\rho = \sqrt{\frac{0.25}{0.75}} = 0.5774$$

The risk-neutral probability that the asset price will be below the critical asset price at time t, $N_1(0.6568)$ is 74.44%. The risk-neutral probability that the asset price will be below S_t^* at time t and below the exercise price X at time T, $N_2(0.2771,0.6568;0.5774)$ is 53.16%. The value of a put on a put with the same terms as the call on the put is 4.0866.

The OPTVAL Function Library contains the compound option valuation function

OV_NS_COMPOUND_OPTION(*s, cp1, x1, tim1, cp2, x2, tim2, r, i, v*)

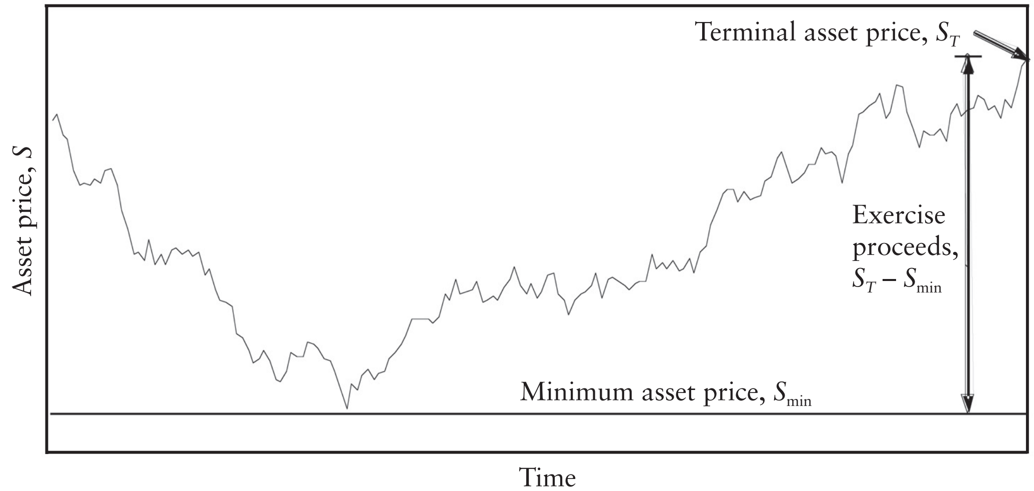
where *cp1* is (c)all/(p)ut indicator for the initial option, *x1* is the exercise price of the initial option, *tim1* is the time to expiration of the initial option, *cp2, x2,* and *tim2* have the same definitions as before except apply to the option delivered if the initial option is exercised, and all other notation is as defined earlier. Thus

OV_NS_COMPOUND_OPTION(1150, "c", 40, 0.25, "p", 1200, 0.75, 0.03, 0.01, 0.15) = 41.6110

LOOKBACK OPTIONS

Aside from compound options and options on the maximum and the minimum, many other exotic options trade in OTC markets. Some of the options are backward looking. A *lookback call option*, for example, provides its holder with settlement proceeds equal to the difference between the terminal asset price and the lowest asset price observed during the life of the option, as is shown in Figure 8.10. A *lookback put option* provides its holder with settlement proceeds equal to the difference between the highest asset price during the life of the option and

FIGURE 8.10 Terminal payoff of lookback call option with a floating exercise price.



the terminal asset price.[14] It should come as no surprise, therefore, that these options are sometimes referred to as "no-regret options."

In a sense, lookback options are like American-style options because the option holder is guaranteed the most advantageous exercise price. Unlike American-style options, however, lookback options can be valued analytically using the BSM risk-neutral valuation mechanics. The reason for this is that it never pays to exercise a lookback option prior to expiration. Independent of how low the exercise price (asset price) has been set thus far during the call option's life, there is always some positive probability that it will fall further. For this reason, the call option holder will always defer early exercise in the hope of recognizing higher exercise proceeds in the future.

Under the assumptions of risk-neutral valuation and lognormally distributed future asset prices, the value of a lookback call may be written as

$$\begin{aligned} c_{\text{lookback}}(S, S_{\min}) = {} & Se^{-iT}N_1(d_1) - S_{\min}e^{-rT}N_1(d_2) \\ & + \frac{S}{\lambda}\left[e^{-rT}\left(\frac{S}{S_{\min}}\right)^{-\lambda} N_1(d_3) - e^{-iT}N_1(-d_1)\right] \end{aligned} \tag{8.42}$$

where S_{min} is the current minimum asset price observed during the option's life,

[14] Many variations of lookback options exist. For a partial summary, see Haug (1998, pp. 61–69). The lookback options discussed in this section have a floating exercise price and were originally valued by Goldman, Sosin, and Gatto (1979). Other lookback options have a fixed exercise and have a terminal payoff equal to the difference between the maximum observed asset price during the option's life and the exercise price in the case of a call and the difference between the exercise price and the minimum observed asset price in the case of a put. These are valued in Conze and Viswanathan (1991).

$$\lambda = \frac{2(r-i)}{\sigma^2}, \; d_1 = \frac{\ln(Se^{-iT}/S_{\min}e^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}, \text{ and } d_3 = -d_1 + \frac{2(r-i)\sqrt{T}}{\sigma}$$

Note that the first two terms of the option are the value of a European-style call option whose exercise price is the current minimum value of the underlying asset. This is the least the lookback call can be worth since the asset price may fall below *X*, thereby driving the exercise price down further.

ILLUSTRATION 8.12 Value lookback call.

Compute the value at inception of a lookback call option that provides its holder with the right to buy the S&P 500 index at any time during the next three months. The S&P 500 index is currently at a level of 1,050, pays dividends at a constant rate of 1% annually, and has a volatility rate of 20%. The risk-free rate of interest is 3%.

The value of the lookback call is therefore

$$\begin{aligned} c_{\text{lookback}}(1{,}050, 1{,}050) &= 1{,}050e^{-0.01(0.25)}N_1(d_1) - 1{,}050e^{-0.03(0.25)}N_1(d_2) \\ &\quad + \frac{1{,}050}{1.000}\left[e^{-0.03(0.25)}\left(\frac{1{,}050}{1{,}050}\right)^{-1.000}N_1(d_3)\right. \\ &\quad \left. - e^{-0.01(0.25)}N_1(-d_1)\right] \\ &= 84.871 \end{aligned}$$

where

$$\lambda = \frac{2(0.03-0.01)}{(0.20)^2} = 1.000, \; d_1 = \frac{\ln(1{,}050e^{-0.01(0.25)}/1{,}050e^{-0.03(0.25)})}{0.20\sqrt{0.25}} = 0.1000$$

$$d_2 = 0.1000 - 0.20\sqrt{0.25} = 0.0000, \text{ and}$$

$$d_3 = -0.1000 + \frac{2(0.03-0.01)\sqrt{0.25}}{0.20} = 0.0000$$

Note that the price of the lookback call is considerably higher than an at-the-money index call option. The value of a European-style call (i.e., the sum of the first two terms in the valuation equation) is only 44.327. The value of a lookback call option can be computed with the OPTVAL function

OV_NS_LOOKBACK_OPTION(*s, sm, t, r, i, v, cp*)

where *sm* is the current minimum asset price for a call (or current maximum price for a put), and the other notation is as defined as before. Thus

OV_NS_LOOKBACK_OPTION(1050, 1050, 0.25, 0.03, 0.01, 0.20, “c”) = 84.430

The value of a lookback put option is

$$p_{LB} = Xe^{-rT}N_1(-d_2) - Se^{-iT}N_1(-d_1) + \frac{S}{\lambda}\left[e^{-iT}N_1(d_1) - e^{-rT}\left(\frac{S}{S_{\max}}\right)^{-\lambda}N_1(-d_3)\right] \tag{8.43}$$

where all notation is as defined for the lookback call except that $S_{\max}$, the current maximum asset price observed during the option's life, replaces $S_{\min}$ in the expression

$$d_1 = \frac{\ln(Se^{-iT}/S_{\max}e^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

Note that a European-style put option is the lower bound for the price of the lookback put option. The third term is necessarily positive. Using the same parameters as in Illustration 8.12, the value of a lookback put option is 83.430, with the underlying standard European-style put being valued at 39.103.

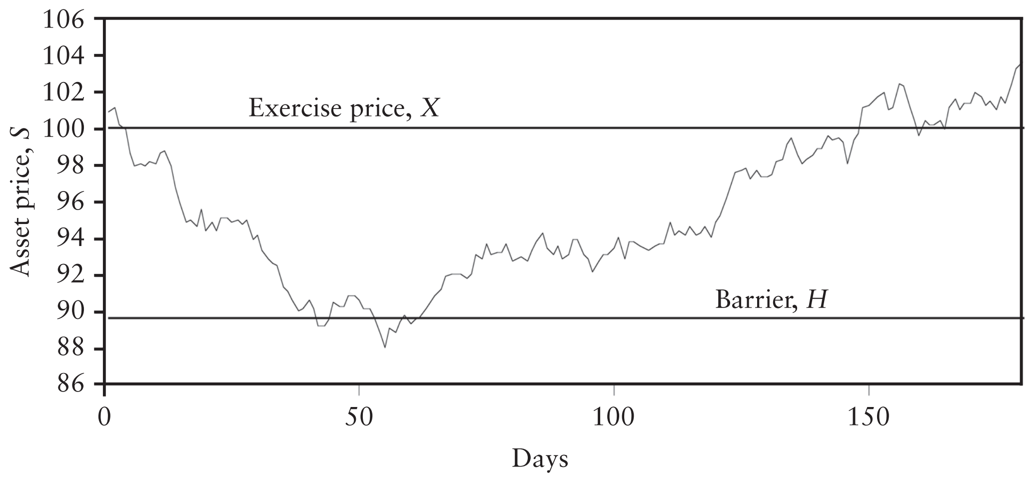
Other backward-looking options are also traded. *Average price* or *Asian options* are based on the average (either arithmetic or geometric) asset price during the option's life. The average asset price may be used as the exercise price of the option, in which case the settlement value of the call will be the terminal asset price less the average price, or it may be used as the terminal asset price, in which case the settlement value will be the average price less the exercise price. Unfortunately, most traded Asian options do not have closed-form valuation equations, and valuation requires the use of numerical methods. Valuing options numerically is the focus of the next chapter.

BARRIER OPTIONS

Barrier options are options that come into existence or terminate automatically when underlying asset price touches a prespecified level.[15] A *down-and-out call*, for example, is a call that expires if the asset price falls below a prespecified "out" barrier, H. At that time, the option buyer may receive a cash rebate, R. A *down-and-in call* is a call that comes into existence if the asset price falls below the "in" barrier, H, at any time during the option's life. For such options, the rebate is received if the option has not knocked in during its lifetime. Figure 8.11 shows a random price path of an asset over a 180-day. If the option has an exercise price of 100 and a barrier of 90, a down-and-out call would cease to exist and a down-and-in call would come into existence on day 39 when the asset price touches 90. Note that if we buy a down-and-out call and a down-and-in call with the same barrier price, H, exercise price, X, and time to expira-

[15] Double barrier options have an upper and lower barrier on the asset price. Their valuation is addressed in in Ikeda and Kunitoma (1992) and Geman and Yor (1996).

FIGURE 8.11 Underlying asset price path for 180-day barrier option with an exercise price of 100 and a barrier of 90.



tion, T, and no rebate, the portfolio has the same payoff contingencies as a standard call option.

The valuation of barrier options is made tedious by the shear number of possible contract specifications.[16] For options with "out" barriers, there are both down-and-out calls and puts. While on face appearance, this would seem to indicate that the number of valuation equations is four. Unfortunately, the number is actually eight since there are two equations for each "out" option, depending on whether the barrier price, H, is above or below the exercise price, X. For options with "in" barriers, the same situation arises, so the total number of equations is 16. To make the presentation of these results as palatable as possible, we adopt the mechanics used in Rubinstein and Reiner (1991). Table 8.1 defines a number of expressions that are used in the valuation equations. Table 8.2 then assembles the valuation equation for each of the 16 different valuation problems.

To illustrate the mechanics, we will value a down-and-out call option whose barrier price, H, is less than the exercise price. As Table 8.2 shows, such an option has a value equal to the first equation in Table 8.1 less the third equation plus the sixth equation (i.e., [1] – [3] + [6]). Piecing things together, we get

$$\begin{aligned} c_{do} = {} & Se^{-iT}N(a_1) - Xe^{-rT}N(a_2) \\ & - Se^{-iT}(H/S)^{2(\mu+1)}N(b_1) + Xe^{-rT}(H/S)^{2\mu}N(b_2) \\ & + R[(H/S)^{\mu+\lambda}N(f_1) + (H/S)^{\mu-\lambda}N(f_2)] \end{aligned} \tag{8.44}$$

[16] A number of authors have focused on barrier options. Merton (1973), for example, values a down-and-out call. Perhaps the most comprehensive treatment is in Rubinstein and Reiner (1991a). This section is based largely of their work. Haug (1998, pp. 65–85) provides valuation procedures for a variety of more complex barrier options.

TABLE 8.1 Definitions required for valuing European-style barrier options. Option notation: X is the exercise price, T is the time to expiration, H is the barrier level, and R is the amount of the cash rebate. Asset notation: S is the asset price, i is the asset's income rate, and σ is the asset's return volatility rate. Other notation: r is the risk-free rate of interest.

[1] $\equiv \phi S e^{-iT} N(\phi a_1) - \phi X e^{-rT} N(\phi a_2)$

[2] $\equiv \phi S e^{-iT} N(\phi b_1) - \phi X e^{-rT} N(\phi b_2)$

[3] $\equiv \phi S e^{-iT} (H/S)^{2(\mu+1)} N(\eta c_1) - \phi X e^{-rT} (H/S)^{2\mu} N(\eta c_2)$

[4] $\equiv \phi S e^{-iT} (H/S)^{2(\mu+1)} N(\eta d_1) - \phi X e^{-rT} (H/S)^{2\mu} N(\eta d_2)$

[5] $\equiv e^{-rT} R[N(\eta b_2) - (H/S)^{2\mu} N(\eta d_2)]$

[6] $\equiv R[(H/S)^{\mu+\lambda} N(\eta f_1) + (H/S)^{\mu-\lambda} N(\eta f_2)]$

where

$$a_1 = \frac{\ln(S/X)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},\ a_2 = a_1 - \sigma\sqrt{T}$$

$$b_1 = \frac{\ln(S/H)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},\ b_2 = b_1 - \sigma\sqrt{T}$$

$$c_1 = \frac{\ln(H^2/SX)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},\ c_2 = c_1 - \sigma\sqrt{T}$$

$$d_1 = \frac{\ln(H/S)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},\ d_2 = d_1 - \sigma\sqrt{T}$$

$$f_1 = \frac{\ln(H/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T},\ f_2 = f_1 - 2\lambda\sigma\sqrt{T}$$

$$\mu = \frac{r - i - \sigma^2/2}{\sigma^2},\ \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}}$$

where H is the barrier asset price below which the call option life ends,

$$a_1 = \frac{\ln(S/X)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},\ a_2 = a_1 - \sigma\sqrt{T}$$

$$c_1 = \frac{\ln(H^2/SX)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},\ c_2 = c_1 - \sigma\sqrt{T}$$

TABLE 8.2 Equations for valuing European-style barrier options. Option notation: X is the exercise price, T is the time to expiration, H is the barrier level, and R is the amount of the cash rebate. Asset notation: S is the asset price.

Description	Condition	Payoff	Valuation equation	Parameters h	Parameters f
Down-and-out call	$S > H$	$\max(S - X,0)$ if $S > H$ before T else R at hit	If $X > H$, [1] – [3] + [6]	1	1
			If $X < H$, [2] – [4] + [6]	1	1
Up-and-out call	$S < H$	$\max(S - X,0)$ if $S < H$ before T else R at hit	If $X > H$, [6]	–1	1
			If $X < H$, [1] – [2] + [3] – [4] + [6]	–1	1
Down-and-out put	$S > H$	$\max(X - S,0)$ if $S > H$ before T else R at hit	If $X > H$, [1] – [2] + [3] – [4] + [6]	1	–1
			If $X < H$, [6]	1	–1
Up-and-out put	$S < H$	$\max(X - S,0)$ if $S < H$ before T else R at hit	If $X > H$, [2] – [4] + [6]	–1	–1
			If $X < H$, [1] – [3] + [6]	–1	–1
Down-and-in call	$S > H$	$\max(S - X,0)$ if $S < H$ before T else R at expiry	If $X > H$, [3] + [5]	1	1
			If $X < H$, [1] – [2] + [4] + [5]	1	1
Up-and-in call	$S < H$	$\max(S - X,0)$ if $S > H$ before T else R at expiry	If $X > H$, [1] + [5]	–1	1
			If $X < H$, [2] – [3] + [4] + [5]	–1	1
Down-and-in put	$S > H$	$\max(X - S,0)$ if $S < H$ before T else R at expiry	If $X > H$, [2] – [3] + [4] + [5]	1	–1
			If $X < H$, [1] + [5]	1	–1
Up-and-in put	$S < H$	$\max(X - S,0)$ if $S > H$ before T else R at expiry	If $X > H$, [1] – [2] + [4] + [5]	–1	–1
			If $X < H$, [3] + [5]	–1	–1

$$f_1 = \frac{\ln(H/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T},\ f_2 = f_1 - 2\lambda\sigma\sqrt{T}$$

$$\text{and } \mu = \frac{r - i - 0.5\sigma^2}{\sigma^2}$$

Note that the sum of the first two terms on the right-hand side, (i.e., [1]), is a standard European-style call option. The value of the standard call is deflated then due to the fact that the option expires automatically when the barrier is touched (i.e., [3]). The last term reflects the potential of receiving a cash rebate (i.e., [6]).

ILLUSTRATION 8.13 Value down-and-out call.

Consider a down-and-out call option with an exercise price of 100, a barrier of 90, no rebate, and a time to expiration of six months. The option's underlying stock has a price of 100, a dividend yield rate of 2%, and a volatility rate of 35%. Compute the value of the down-and-out call assuming the risk-free interest rate is 4%.

From Table 8.2, we know the value of the down-and-out call with no rebate is [1] – [3] and can be written

$$\begin{aligned} c_{do} &= 100e^{-0.02(0.5)}N(a_1) - 100e^{-0.04(0.5)}N(a_1 - 0.35\sqrt{0.5}) \\ &\quad - 100e^{-0.02(0.5)}(90/100)^{2(\mu+1)}N(b_1) \\ &\quad + 100e^{-0.04(0.5)}(90/100)^{2\mu}N(b_1 - 0.35\sqrt{0.5}) \\ &= 7.4378 \end{aligned}$$

where

$$a_1 = \frac{\ln(100/100)}{0.35\sqrt{0.5}} + (1 - 0.3367)0.35\sqrt{0.5} = 0.1641,$$

$$a_2 = 0.1641 - 0.35\sqrt{0.5} = -0.0833$$

$$c_1 = \frac{\ln(90^2/(100 \times 100))}{0.35\sqrt{0.5}} + (1 - 0.3367)0.35\sqrt{0.5} = -0.6873$$

$$c_2 = -0.6873 - 0.35\sqrt{0.5} = -0.9348$$

$$\text{and } \mu = \frac{0.04 - 0.02 - 0.5(0.35^2)}{0.35^2} = -0.3367$$

This value can be verified using the OPTVAL function,

OV_NS_BARRIER_OPTION(*s, x, h, t, rebate, r, i, v, TypeFlag*)

where *s* is the asset price, *x* is the exercise price, *h* is the barrier level, *t* is the time to expiration, *r* is the risk-free rate of interest, *i* is the income rate, and *v* is the volatility rate. The *TypeFlag* consists of three contiguous lower case letters. The first is a (c)all/(p)ut indicator, the second is a (d)own/(u) indicator, and the third is a (i)n/(o)ut indicator. For a down-and-out call, *TypeFlag* is "cdo." Hence, the value of a down-and-out call is

OV_NS_BARRIER_OPTION(100, 100, 90, 0.5, 0, 0.04, 0.02, 0.35, "cdo") = 7.4378

Hence, the value of a down-and-in call is

OV_NS_BARRIER_OPTION(100, 100, 90, 0.5, 0, 0.04, 0.02, 0.35, "cdi") = 2.7643

The sum of the values of the down-and-out call and the down-and-in call (with no rebate) equals the value of a standard European-style call option, that is,

OV_OPTION_VALUE (100, 100, 0.5, 0.04, 0.02, 0.35, "c", "e") = 10.2021

SUMMARY

This chapter focuses on the valuation of some nonstandard option contracts traded in the OTC market. One characteristic shared by all of these options are that they are valued under the BSM risk-neutral, lognormal asset price distribution framework. Another is that they have analytical valuation equations. As such, the options included in this chapter are European-style. (The valuation of American-style nonstandard options requires the use of numerical methods, which is the focus of the next chapter.) Interestingly, most of the options can be valued using valuation-by-replication and the valuation results derived in Chapter 5. The options valued in this chapter include:

- All-or-nothing options
- Gap options
- Contingent pay options
- Forward-start options
- Ratchet options
- Chooser options
- Exchange options
- Options on the maximum and the minimum
- Compound options
- Lookback options
- Barrier options

While this list seems to cover a wide range of nonstandard option contracts, do not be misled—we have only discussed eleven of a countless number of option contract designs that exist in the OTC markets. Others will be discussed as the chapters progress in the chapters that follow. Whether an option is valued analytically or numerically is of no consequence to risk measurement. The risk characteristics of options can be measured numerically with a high degree of accuracy.

REFERENCES AND SUGGESTED READINGS

Abramowitz, Milton, and Irene A. Stegum. 1972. *Handbook of Mathematical Functions*, 10th ed. Washington, DC: National Bureau of Standards.

Black, Fischer, and Myron Scholes. 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81: 637–659.
Conze, A., and R. Viswanathan. 1991. Path dependent options: The case of lookback options. *Journal of Finance* 46: 1893–1907.
Drezner, Z. 1978. Computation of the bivariate normal integral. *Mathematics of Computation* 32: 277–279.
Geman, H., and M. Yor. 1996. Pricing and hedging double barrier options. *Mathematical Finance* 6: 365–378.
Geske, Robert. 1979. The valuation of compound options. *Journal of Financial Economics* 7: 63–81.
Goldman, B., H. Sosin, and M. A. Gatto. 1979. Path dependent options: Buy at the low, sell at the high. *Journal of Finance* 34 (December): 1111–1127.
Haug, Espen Gaarder. 1998. *The Complete Guide to Option Pricing Formulas*. New York: McGraw-Hill.
Ikeda, M., and N. Kunitomo. 1992. Pricing options with curved boundaries. *Mathematical Finance* 2: 275–298.
Johnson, Herbert E. 1987. Options on the minimum and maximum of several assets. *Journal of Financial and Quantitative Analysis* 22 (September): 277–283.
Margrabe, William. 1978. The value of an option to exchange one asset for another. *Journal of Finance* 33 (March): 177–186.
Margrabe, William. 1982. A theory of the price of a contingent claim on N asset prices. Working paper 8210 (September 1982), School of Government and Business Administration, George Washington University.
Merton, Robert C. 1973. Theory of rational option pricing. *Bell Journal of Economics and Management Science*: 141–183.
Rubinstein, Mark. 1991a. Pay now, choose later. *RISK* 4 (February): 13.
Rubinstein, Mark. 1991b. Options for the undecided. *RISK* 4 (April): 43.
Rubinstein, Mark, and Eric Reiner. 1991a. Breaking down the barriers. *RISK* 4 (September): 31–35.
Rubinstein, Mark, and Eric Reiner. 1991b. Unscrambling the binary code. *RISK* 4 (October): 75–83.
Stulz, Rene. 1982. Options on the minimum or maximum of two assets. *Journal of Financial Economics* 10: 161–185.

APPENDIX 8A: APPROXIMATION OF THE BIVARIATE NORMAL PROBABILITY

The joint probability that x is less than a and y is less than b is

$$\begin{aligned}\Pr(x \le a, y < b) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{a}\int_{-\infty}^{b} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right] dx\,dy \\ &= N_2(a, b, \rho)\end{aligned}$$

where x and y are random variables with unit normal distributions (i.e., mean 0 and variance 1) with correlation, ρ. The approximation method provided here relies on Gaussian quadratures,[17] and has a maximum absolute error of 0.00000055.

First, program a routine that evaluates the term $\phi(a,b;\rho)$.

$$\phi(a, b;\rho) \approx 0.31830989\sqrt{1-\rho^2}\sum_{i=1}^{5}\sum_{j=1}^{5} w_i w_j f(x_i, x_j)$$

where

$$f(x_i, x_j) = \exp[a_1(2x_i - a_1) + b_1(2x_j - b_1) + 2\rho(x_i - a_1)(x_j - b_1)]$$

the pairs of weights (w) and the corresponding abscissa values (x) are:

i,j	***w***	***x***
1	0.24840615	0.10024215
2	0.39233107	0.48281397
3	0.21141819	1.0609498
4	0.03324666	1.7797294
5	0.000824853	2.6697604

and the coefficients are computed using

$$a_1 = \frac{a}{\sqrt{2(1-\rho^2)}} \quad \text{and} \quad b_1 = \frac{b}{\sqrt{2(1-\rho^2)}}$$

Next, compute the product, $ab\rho = a \times b \times \rho$. If $ab\rho \le 0$, compute the bivariate normal probability by applying one of the following rules:

If			**Then**
$a \le 0$	$b \le 0$	$\rho \le 0$	$N_2(a,b;\rho) = \phi(a,b;\rho)$
$a \le 0$	$b \ge 0$	$\rho \ge 0$	$N_2(a,b;\rho) = N_1(a) - \phi(a,-b;-\rho)$
$a \ge 0$	$b \le 0$	$\rho \ge 0$	$N_2(a,b;\rho) = N_1(b) - \phi(-a,b;\rho)$
$a \ge 0$	$b \ge 0$	$\rho \le 0$	$N_2(a,b;\rho) = N_1(a) + N_1(b) - 1 + \phi(-a,-b;\rho)$

where $N_1(d)$ is the cumulative univariate normal probability. (Recall that Appendix 7C in the previous chapter contains the approximation algorithm for the cumulative univariate normal probability $N_1(d)$. It is also available as the function OV_PROB_PRUN(a) in the OPTVAL Function Library.) If $ab\rho > 0$, compute the bivariate normal probability as

$$N_2(a, b;\rho) = N_2(a, 0;\rho_{ab}) + N_2(b, 0;\rho_{ba}) - \delta$$

where the values of $N_2(\cdot)$ on the right-hand side are computed using the rules for $ab\rho \le 0$,

[17] The Gaussian quadrature method for approximating the bivariate normal probability is from Drezner (1978).

$$\rho_{ab} = \frac{(\rho a - b)\mathrm{Sgn}(a)}{\sqrt{a^2 - 2\rho ab + b^2}}, \quad \rho_{ba} = \frac{(\rho b - a)\mathrm{Sgn}(b)}{\sqrt{a^2 - 2\rho ab + b^2}}$$

$$\delta = \frac{1 - \mathrm{Sgn}(a) \times \mathrm{Sgn}(b)}{4}, \text{ and}$$

$$\mathrm{Sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Applying this procedure provides the following probabilities:

a	b	ρ	$N_2(a,b;\rho)$
–1	–1	–0.5	0.00378
–1	1	–0.5	0.09614
1	–1	–0.5	0.09614
1	1	–0.5	0.68647
–1	–1	0.5	0.06251
–1	1	0.5	0.15487
1	–1	0.5	0.15487
1	1	0.5	0.74520
0	0	0.5	0.33333
0	0	0	0.25000
0	0	–0.5	0.16667

To check these values, you may use OV_PROB_PRBN from the OPTVAL Function Library.