CHAPTER 3

Relation between Return and Risk

An important facet of valuation not yet discussed is the relation between expected return and risk. In the bond and stock valuation models discussed in Chapter 2, risk enters into the valuation formulas through the interest rate used to discount the expected cash flows to the present. In financial economics, the *capital asset pricing model* (CAPM)[1] provides the structural relation between expected return and risk. It relies on the assumption that individuals prefer more wealth to less wealth, but at a decreasing rate. Such individuals are risk averse, and risk aversion is the focus of the utility theory discussion in the first section. In the second section, we extend the discussion to show how such individuals allocate their wealth among securities. In the third section, we aggregate security demands across all individuals in the marketplace and identify the equilibrium expected return/risk relations for individual securities and security portfolios. Finally, in the fourth section, we apply the CAPM relations to evaluate portfolio performance.

UTILITY THEORY

In most financial economic models, individuals are assumed to be *risk averse*. Investors do not like risk but are willing to bear it if paid an adequate risk premium. Risk premiums arise from the nature of how an individual's satisfaction varies with wealth. Called a *utility of wealth function*, $U(w)$ is the level of satisfaction (measured in units of utility) realized from having a level of wealth, w. An individual's marginal utility of wealth is assumed to be positive (i.e., $dU(w)/dw > 0$)—the more wealth, the more satisfaction. Indeed, this property is the driving force behind the absence of costless arbitrage opportunities in a rationally functioning marketplace. As wealth increases, however, the rate at which satisfaction increases falls (i.e., $d^2U(w)/dw^2 > 0$). The next dollar earned is not quite as satisfying as the last dollar earned.

[1] The central role that the CAPM plays in financial economics is attested to by the fact that five of the key players in its development—Harry Markowitz, James Tobin, William Sharpe, John Lintner, and Robert C. Merton—have received Nobel Prizes in Economics.

Figure 3.1 illustrates the shape of the utility function for an individual with diminishing positive marginal utility of wealth. Note that, as wealth increases, utility increases, but at a decreasing rate. To show that this individual is a risk averter, consider his behavior when presented with a fair bet. A *fair bet* is any bet whose expected outcome is 0. A 50-50 chance of winning or losing X, for example, constitutes a fair bet. Accepting a fair bet implies that there is no change in the individual's expected wealth level. If the individual's certain wealth level before the bet is w_0, his expected wealth level upon accepting the bet remains at w_0, that is,

$$E(\tilde{w}) = 0.5(w_0 + \tilde{X}) + 0.5(w_0 - \tilde{X}) = w_0$$

But, because the individual's expected wealth does not change, that does not mean he is indifferent about whether or not to accept the bet. He will not. The reason is that, after accepting the bet, his expected satisfaction level is

$$E[\tilde{U}(w)] = 0.5U(w_0 + \tilde{X}) + 0.5U(w_0 - \tilde{X})$$

Because his utility function is concave from below, as shown in Figure 3.2, the expected utility of wealth after taking the bet rests below the utility that he had to begin with, $E[\tilde{U}(w)] < U(w_0)$. Because taking a fair bet reduces the individual's expected utility, an individual with diminishing positive marginal utility of wealth is said to be a *risk averter.*

FIGURE 3.1 Utility function of a risk averse individual.

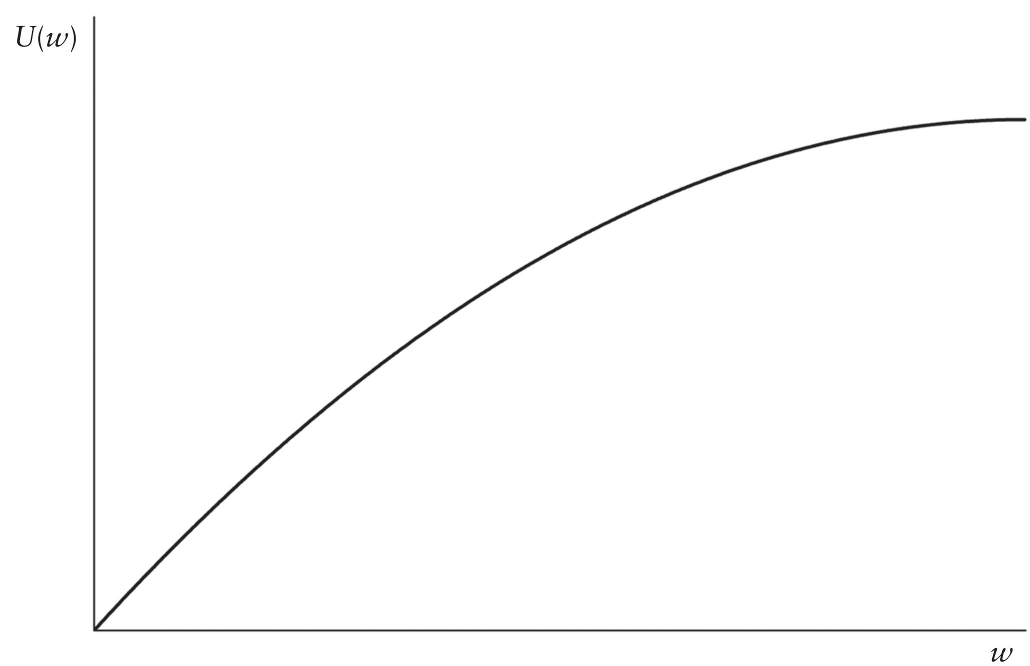
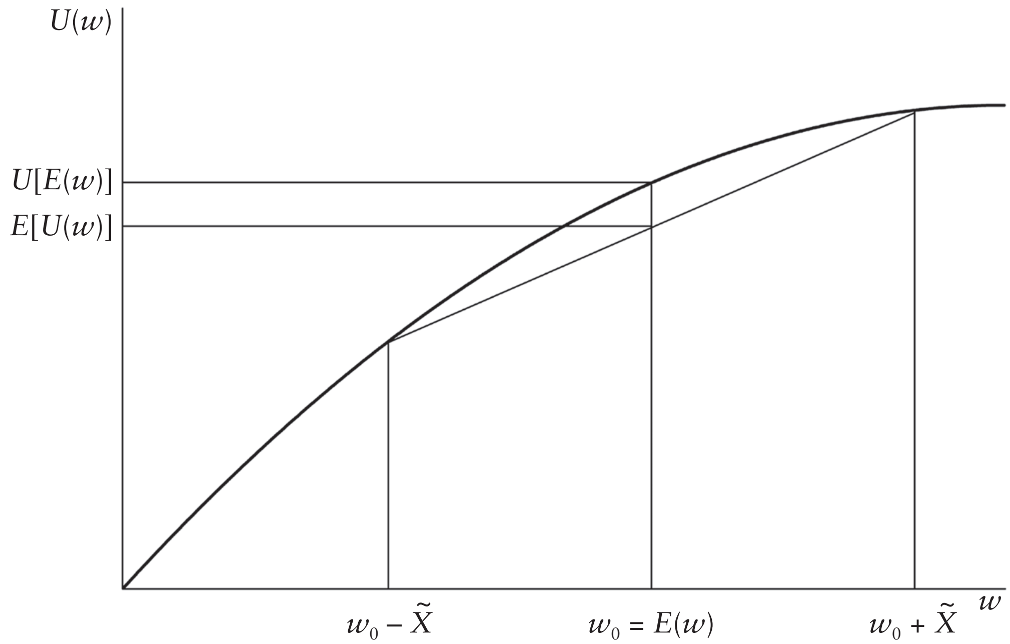


FIGURE 3.2 Utility function of a risk averse individual when evaluating a fair bet.



Utility theory is useful not only in demonstrating general behavioral principles but also in evaluating specific investment opportunities. In order to make specific decisions, however, the mathematical character of the utility function needs to be more precisely defined.[2] A logarithmic utility of wealth function, $U(w) = \ln w$, mimics the behavior of a risk-averter—$dU(w)/dw = 1/w > 0$ and $d^2U(w)/dw^2 = -1/w^2 < 0$. So does a square root utility of wealth function, $U(w) = \sqrt{w}$, since $dU(w)/dw = 0.5w^{-0.5} > 0$ and $d^2U(w)/dw^2 = -0.25w^{-1.5} < 0$. We use these utility of wealth functions in the illustrations that follow.

Aside from knowing the specific character of the utility function, a handful of definitions are also important. In the illustrations that follow, we assume that the individual holds two assets—a risk-free asset whose value is R and a risky asset whose value in one period is either $\tilde{X}_1$ or $\tilde{X}_2$, with probabilities p and $1 - p$, respectively. Assuming the risk-free interest rate is zero, the individual's *expected utility of terminal wealth* is

$$E[\tilde{U}(w)] = pU(R + \tilde{X}_1) + (1-p)U(R + \tilde{X}_2) \tag{3.1}$$

Now, suppose someone approaches this individual and asks him to sell his risky asset. What is the least amount that the individual will take? To answer this question, we must first identify the cash equivalent of the individual's overall

[2] The four most commonly used utility of wealth functions used in financial economics are: (a) the logarithmic utility function $U(w) = \ln w$, (b) the quadratic utility function $U(w) = aw - bw^2$ where $a > 2bw$ and $b > 0$, (c) the exponential utility function $U(w) = -e^{-aw}$ where $a > 0$, and (d) the power utility function $U(w) = w^a$ where $0 < a < 1$.

position. The *cash equivalent* is that certain amount of cash, C, that the individual is willing to take for his entire position and is computed by setting the utility of the cash amount equal to the expected utility of terminal wealth, that is,

$$U(C) = E[\tilde{U}(w)] \tag{3.2}$$

With the amount of the cash equivalent, C, known, the *minimum selling price* of the risky asset equals $C - R$.

ILLUSTRATION 3.1 Identify maximum insurance premium.

Consider two individuals—A with a logarithmic utility function and B with a square root utility function. Both individuals have \$100,000 in cash and face the prospect of losing \$50,000, with a 5% probability. What is the maximum amount that each individual would be willing to pay for insurance?

Individual A currently enjoys an expected satisfaction level,

$$E[\tilde{U}_A(w)] = 0.05\ln(100{,}000 - 50{,}000) + 0.95\ln(100{,}000) = 11.478$$

Holding expected utility constant, this implies that A is indifferent between staying in his current position (i.e., holding \$100,000 in cash and having the prospect of losing \$50,000) and having a certain amount of cash C as determined by

$$U_A(C) = \ln C = 11.478$$

Solving for C, you find $C = e^{11.478} = 96{,}593.63$. In other words, A is indifferent between having (a) \$100,000 in cash and running a 5% chance of losing \$50,000, and (b) \$96,593.63 in cash. Thus, the maximum amount A is willing to pay for insurance against loss is \$100,000 – 96,593.63 or \$3,406.37.

Individual B currently enjoys a satisfaction level,

$$E[\tilde{U}_B(w)] = 0.05\sqrt{100{,}000 - 50{,}000} + 0.95\sqrt{100{,}000} = 311.597$$

Individual B's cash equivalent wealth level is determined by

$$U_B(C) = \sqrt{C} = 311.597$$

Solving for C, $C = 311.597^2 = 97{,}092.51$, which means B is willing to pay up to \$2,907.49 for insurance against loss. Apparently an individual with logarithmic utility is more risk averse than an individual with square root utility.

ILLUSTRATION 3.2 Are options really a zero-sum game?

In Chapter 1, derivative trades are described as zero-sum games—what the buyer gains, the seller loses, and vice versa. This does not *imply, however, that both the buyer and the seller cannot gain from trading. Assume Individual A has \$50 in cash and one share of a common stock. The stock, he believes, has a 60% chance of falling in price to \$80 and a 40% chance of increasing in price to \$120. Individual B has \$100 in cash and no other holdings. B, however, follows the stock held by A and is much more optimistic regarding its prospects. Specifically, B assigns only a 30% chance of the stock of falling in price to \$80 and a 70% chance of it increasing to \$120. Demonstrate that both A and B can both be made better of by trading a put option written on the stock. Assume the put has an* *exercise price of $100 and costs $10. Assume both individuals have square root utility functions. Ignore the time value of money.*

Assume Individual A wants to buy the put, considering his pessimistic outlook regarding the stock's prospects. A's current expected utility level is

$$E[\tilde{U}(w_A)] = 0.6\sqrt{50+80} + 0.4\sqrt{50+120} = 12.06$$

where the terminal wealth levels are 130 or 170, depending on the performance of the stock. On the other hand, if he buys the put, his terminal wealth level is 130 less the put price plus the payoff on the put if the stock price falls and is 170 less the put price is the stock price rises. Thus, if he buys the put, his expected utility is

$$E[\tilde{U}(w_A)] = 0.6\sqrt{130-10+(100-80)} + 0.4\sqrt{170-10+0} = 12.16$$

Thus, from an expected utility of terminal wealth standpoint, A is made better off by buying the put.

Individual B, on the other hand, is more optimistic regarding the stock's prospects and is considering selling the put. B currently enjoys a utility of wealth equal to

$$U(w_B) = \sqrt{100} = 10$$

If he sells the put, his terminal wealth will be either 100 plus the put price less the put payoff (that goes to A) if the stock price falls and 100 plus the put price if the stock price rises. Thus, after selling the put, his expected utility of wealth is

$$E[\tilde{U}(w_B)] = 0.3\sqrt{100+10-(100-80)} + 0.7\sqrt{100+10-0} = 10.09$$

Since selling the put provides B a portfolio with higher expected utility of wealth, B, too, is made better off by the trade.

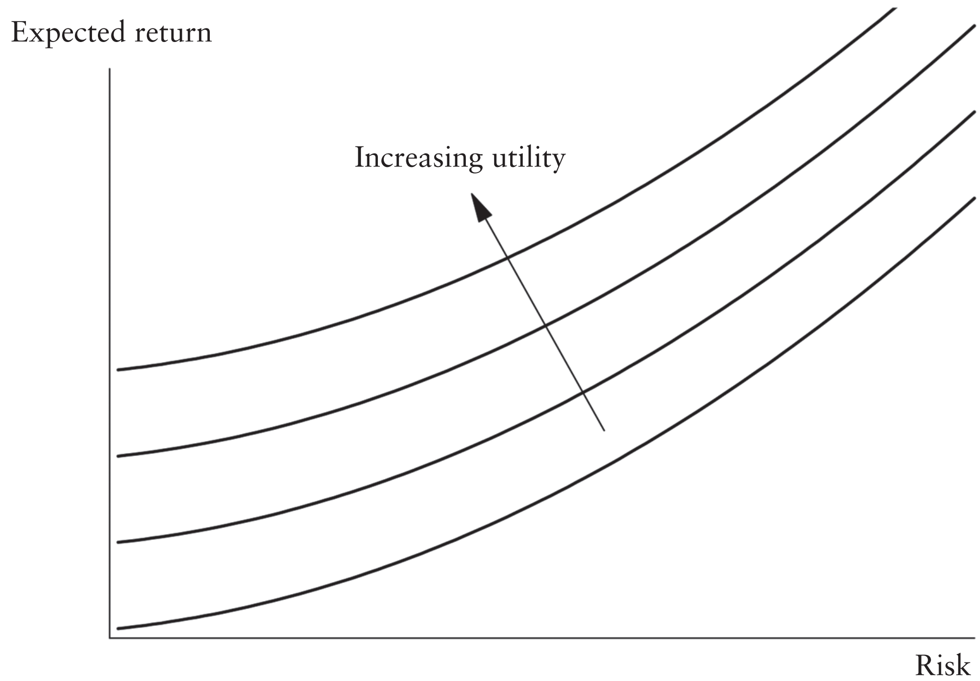
The fact that both A and B are made better off by trading with each other does not negate the fact that the trade, in itself, is zero-sum. It is. If the stock price falls, A's net payoff equals the option payoff less the put price, that is, (100 – 80) – 10 = +10, and B's net payoff is the put price less the exercise proceeds 10 – (100 – 80) = –10. If the stock price rises, the put expires worthless, which means that A's net loss on the put, 10, is B's net gain.

PORTFOLIO THEORY

The expected utility framework is useful in a number of decision-making contexts. A weakness of the framework, however, is that the individual's utility function must be specified. Exactly how one goes about identifying the mathematical structure of an individual's utility function is unclear. Fortunately, for the individual's portfolio allocation decision, a specific structure is not necessary. The reason is that Tobin (1958) shows that individuals with diminishing positive marginal utility have expected return (E)/risk (σ) indifference curves shaped like those shown in Figure 3.3,[3] where risk is measured by the standard deviation of return. Along each

[3] Technically speaking, Tobin (1958) proved this result in two general cases: (a) individuals have quadratic utility of wealth; and (b) the distribution of security returns is multivariate normal.

FIGURE 3.3 Indifference curves of a risk-averse individual.



indifference curve, expected utility is held constant. The curves have the properties $dE/d\sigma > 0$ and $d^2E/d\sigma^2 > 0$. The first derivative says that an individual will demand a more return as risk increases. The second derivative says that the rate at which the individual demands more return grows faster and faster as risk increases. In Figure 3.3, note also that the higher the indifference curve, the greater the expected utility. That means individuals choose portfolios that have the highest expected return for a given level of risk and/or portfolios that have the lowest risk for a given level of expected return. Such portfolios are called *efficient portfolios*.

Prior to formulating the individual's portfolio allocation decision, it is worthwhile to note that how a risk-averter's indifference curves differ from those of an individual who is risk-neutral. Figure 3.4 illustrates the indifference curves of a risk-neutral individual. The fact that the curves are horizontal means that a risk-neutral individual does not care about risk. Such an individual chooses a portfolio that maximizes expected return. At the other behavioral extreme are indifference curves that are vertical, as shown in Figure 3.5. This individual is a *risk minimizer* and will choose a portfolio that minimizes portfolio risk.

Portfolio Allocation with *n* Risky Securities

The focus now turns to identifying efficient portfolios, that is, portfolios with the highest expected return for a given level of risk and/or with the lowest risk for a given level of expected return. To do so, an individual must gather a considerable amount of information. Assuming n risky securities exist in the marketplace, an

FIGURE 3.4 Indifference curves of a risk-neutral individual.

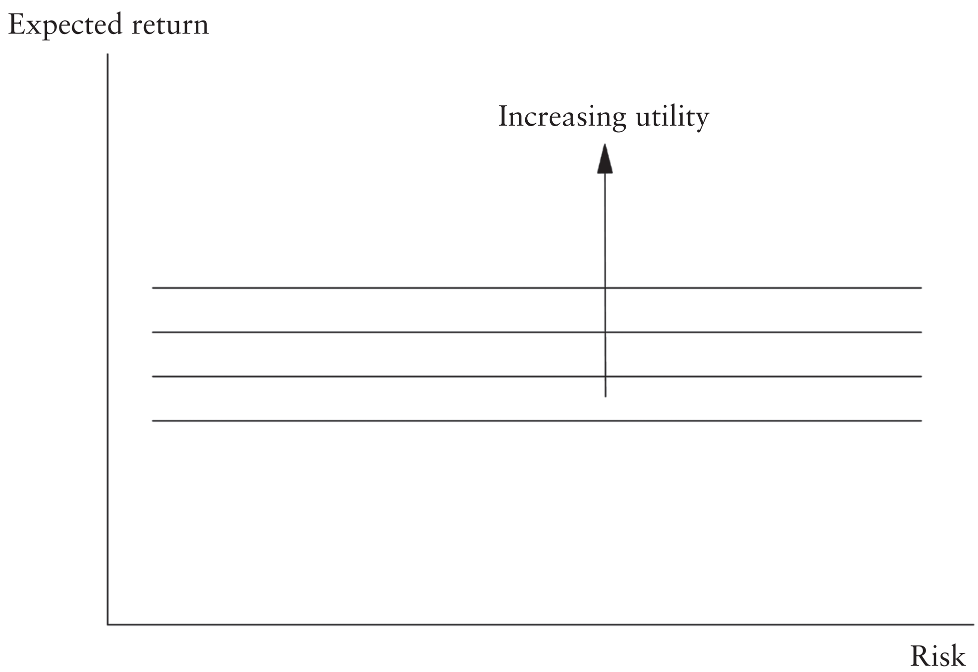
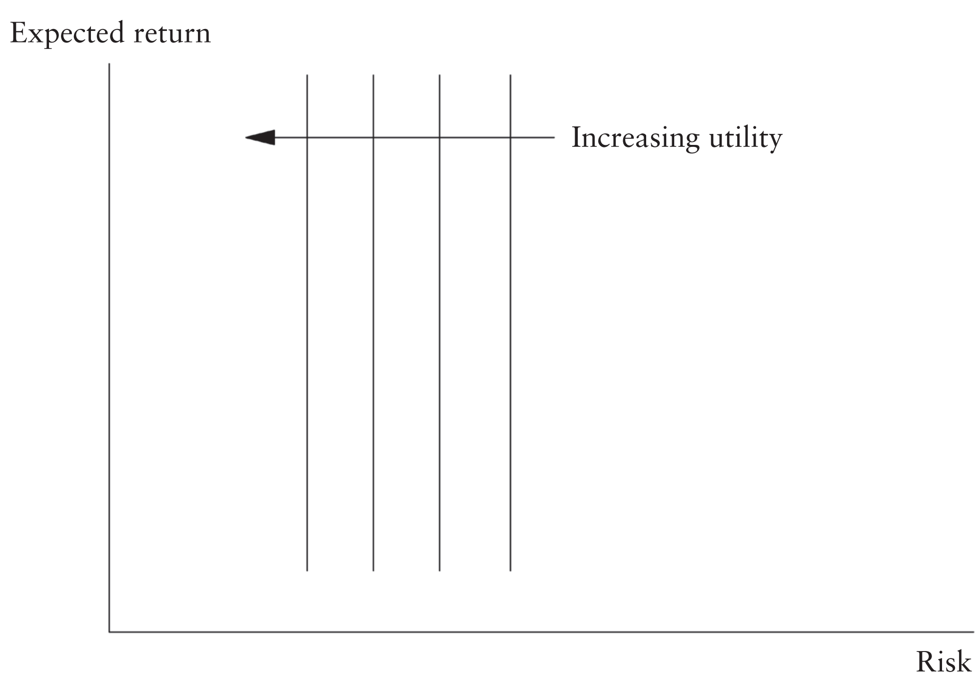


FIGURE 3.5 Indifference curves of a risk minimizer.



individual must estimate: (a) the expected return of each risky security, E_i, $i = 1, \ldots, n$, (b) the standard deviation of return of each risky security, σ_i, $i = 1, \ldots, n$ and (c) the correlation of returns for each pair of securities in the marketplace, ρ_{ij}, $i = 1, \ldots, n$ and $j = 1, \ldots, n$. At first blush, there seems to be a need to estimate n^2 different correlation coefficients. With respect to these correlations, however, we know that $\rho_{ij} = +1$ where $i = j$ and that $\rho_{ij} = \rho_{ji}$. This reduces the number of necessary estimates to $n(n-1)/2$. For expositional convenience, covariances are used below. The covariance between the returns of securities i and j is defined as $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$.

Certain definitions are required to set up the portfolio allocation problem. The expected return on portfolio S is

$$E_S = \sum_{i=1}^{n} X_i E_i \tag{3.3}$$

where X_i is the proportion on the individual's wealth invested in security i. Naturally the sum of the proportions equals 1, that is,

$$\sum_{i=1}^{n} X_i = 1$$

This is sometimes called the *wealth constraint*. The standard deviation of the portfolio return is

$$\sigma_S = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \sigma_{ij}} \tag{3.4}$$

Now, to identify the individual's optimal allocation among the n risky securities, we minimize portfolio risk,

$$\text{Minimize } \sigma_S^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \sigma_{ij} \tag{3.5}$$

subject to

$$\sum_{i=1}^{n} X_i E_i = E_S \tag{3.5a}$$

and

$$\sum_{i=1}^{n} X_i = 1 \tag{3.5b}$$

Constraint (3.5a) requires that the weights produce an expected portfolio return equal to the target level, E_S, and constraint (3.5b) requires that all risky security wealth is fully allocated. The objective function (3.5), together with the constraints (3.5a) and (3.5b), constitute a *nonlinear programming problem*. Some such problems can be solved analytically; other numerically. For current purposes, however, it is sufficient to know that, as long as no two risky securities have returns that are perfectly correlated, the solution to the problem is a unique set of allocations, X_i^*, $i = 1, \ldots, n$, that produce a minimum variance portfolio. If we solve this portfolio allocation problem for a range of levels of target expected portfolio return, E_S, we can trace out the minimum variance (or minimum risk) frontier shown in Figure 3.6. This frontier is sometimes referred to as the *Markowitz* (1952) *efficiency frontier*, in honor of Nobel laureate, Harry Markowitz, who originally developed the framework more than 50 years ago.

FIGURE 3.6 Minimum variance (or Markowitz efficiency) frontier.

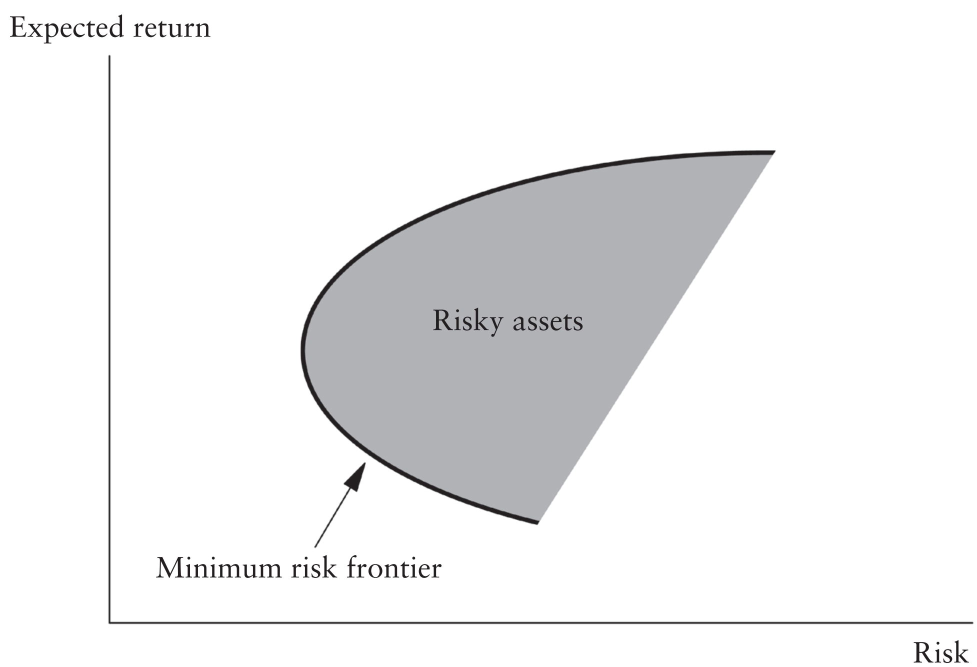


ILLUSTRATION 3.3 Identify efficient portfolios comprised of two risky securities.

Describe the range of efficient portfolio allocations when only two risky securities are available in the marketplace. The expected returns and standard deviations of returns of the two securities are shown below. Assume the correlation between the returns of security 1 and security 2 is 0.25.

Security	Expected Return	Standard Deviation
1	18%	20%
2	12%	16%

In order to identify the set of efficient portfolios that can be generated by allocating your wealth between securities 1 and 2, you first need to identify what portfolios are *feasible*. The expected return and standard deviation of return of portfolios created by allocating wealth between security 1 and security 2 are given by (3.3) and (3.4), where the number of securities n equals 2. Thus, the expected portfolio return is

$$\begin{aligned} E_S &= X_1 E_1 + (1 - X_1) E_2 \\ &= X_1(0.18) + (1 - X_1)(0.12) \end{aligned}$$

and the standard deviation of portfolio return is

$$\begin{aligned} \sigma_S &= \sqrt{X_1^2 \sigma_1^2 + 2X_1(1 - X_1)\rho_{12}\sigma_1\sigma_2 + (1 - X_1)^2 \sigma_2^2} \\ &= \sqrt{X_1^2(0.20)^2 + 2X_1(1 - X_1)(0.25)(0.20)(0.16) + (1 - X_1)^2(0.16)^2} \end{aligned}$$

Note that since there are only two securities, the proportion of wealth invested in security 2 is $X_2 = 1 - X_1$. The rest of the exercise is a matter of computing E_S and σ_S for different levels of X_1. This can be easily accomplished in Microsoft Excel. The table below summarizes the results.

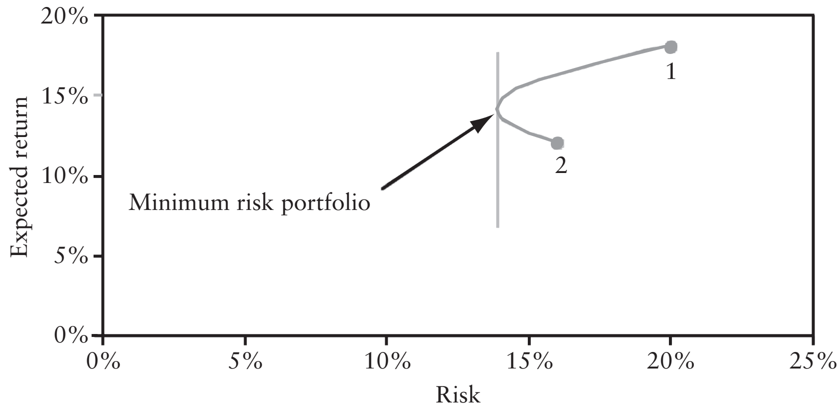
Proportion of Wealth Invested in Security		**Portfolio Attributes**	
1	**2**	**Expected Return**	**Standard Deviation**
1.0	0.0	18.00%	20.00%
0.9	0.1	17.40%	18.47%
0.8	0.2	16.80%	17.08%
0.7	0.3	16.20%	15.89%
0.6	0.4	15.60%	14.95%
0.5	0.5	15.00%	14.28%
0.4	0.6	14.40%	13.95%
0.3	0.7	13.80%	13.97%
0.2	0.8	13.20%	14.33%
0.1	0.9	12.60%	15.03%
0.0	1.0	12.00%	16.00%

The table shows that the expected portfolio return falls from 18% to 12% as the proportion of wealth invested in security 1 goes from 1 to 0. The standard deviation of portfolio return, on the other hand, initially falls as X_1 is reduced, but then begins to rise again after X_1 passes the level 0.40 on its way to zero. The figure below summarizes the results. Exactly what allocation produces the minimum risk portfolio can be determined by taking the derivative of the portfolio standard deviation and setting it equal to 0. For the two-security portfolio, the minimum risk allocation is

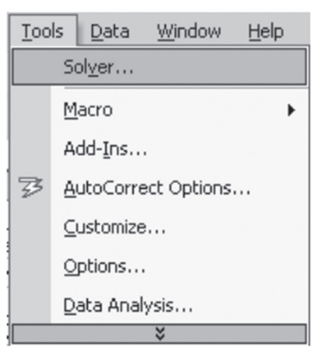
$$X_1 = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2} \tag{3.6}$$

Substituting the problem parameters, we find that the risk-minimizing portfolio is created by allocating 0.355 of wealth to security 1 and 0.645 to security 2. This portfolio has an expected return equal to 14.13% and a standard deviation of return equal to 13.91%. Thus, while the above table shows the range of feasible portfolios that can be created by allocating one's wealth between security 1 and security 2, no risk-averse individual will hold a portfolio with less (more) than 0.355 (0.645) of his wealth allocated to security 1 (2). The range of allocations that produces *efficient portfolios* is $0.355 \le X_1 \le 1$.[4]

A short digression is important here. While we have identified the range of allocations that produces efficient portfolios, one efficient portfolio—the minimum risk portfolio—will *never* be held by a risk averter. The reason is that the slope of the expected return/risk frontier at the minimum risk portfolio is infinite. An individual whose indifference curves have the properties $dE/d\sigma > 0$ and $d^2E/d\sigma^2 > 0$ is not allowed to choose such a portfolio. Consequently, the range of portfolios from which a risk averter chooses his optimal portfolio is defined by $0.355 < X_1 \le 1$.[5]



It is also worthwhile to note the Microsoft Excel has a powerful tool called *Solver* that is useful in nonlinear optimization problems such as this. You will find it in the Tool menu:



The Excel file, **Two-asset portfolio.xls,** contains the expected return/risk and correlation attributes of the two securities used in this illustration. It shows how Solver can be used to identify the risk-minimizing portfolio numerically. Note that in the worksheet below, the portfolio risk appears in cell C16. At the top of the Solver Parameters box, you tell Solver to find a minimum value for C16 by changing the cells C11 and C12. Just below, you tell Solver that the sum of the allocations to the risky securities, C13, must equal one. Click "Solve" and you are done. This solution, computed using a numerical search procedure, is equal to the risk-minimizing allocations that you solved for analytically using (3.6):

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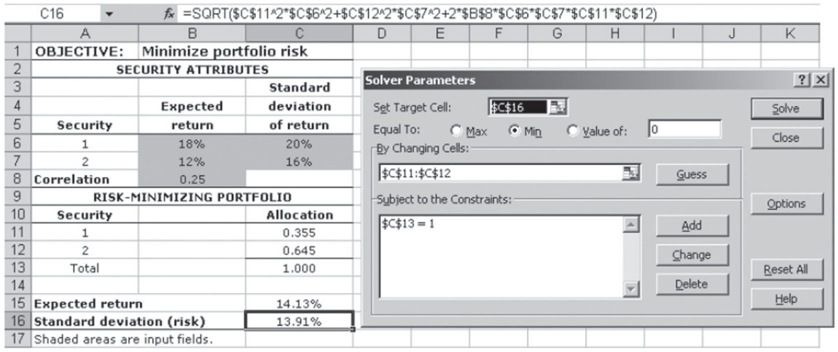
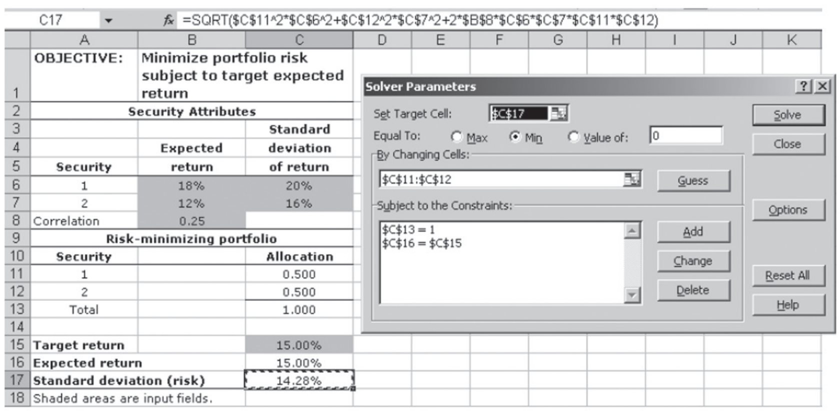
[4] Technically speaking, the range of efficient portfolios continues on to the right of security 1 since short sales of security 2 are permitted.

[5] This distinction becomes very important in the risk management strategies of Chapter 5.



To trace out the entire risky asset efficient frontier, you need to modify the instructions to the Solver tool. Instead of solving for the portfolio weights that unconditionally minimize the risk of the portfolio, suppose you minimize portfolio risk subject to the constraint that the "target" expected return on the portfolio equals, say, 15%. The modified instructions are as follows:



Note that you need to impose the additional constraint that the expected return of the portfolio, C16, equals the target rate of return C15. The risk-minimizing portfolio with an expected return of 15% has equal allocations of wealth in assets 1 and 2.

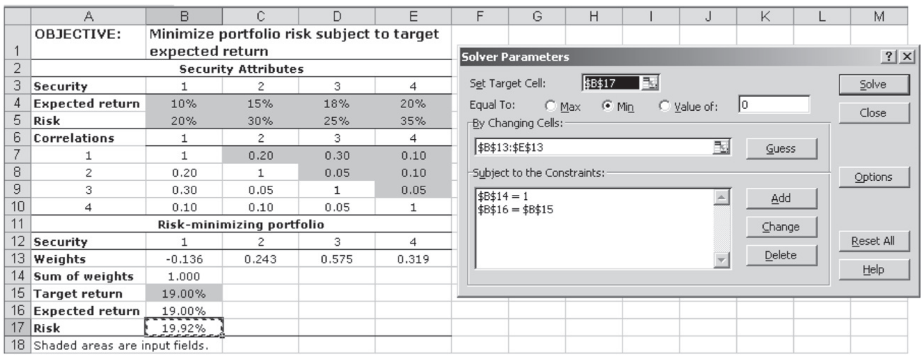
ILLUSTRATION 3.4 Identify efficient portfolios comprised of four risky securities.

Assume you have four risky securities available for investment. Their expected returns, risks, and correlations are presented in the shaded areas of the table below. With four risky securities, the number of required parameter estimates is 14: (a) four expected security returns, (b) four standard deviations of security returns, and (c) $n(n-1)/2 = 4(4-1)/2 = 6$ correlations between pairs of security returns. Find the risk-minimizing portfolio with a target expected return of 19%. Do so first allowing short sales, and then disallowing short sales.

Security Attributes				
Security	**1**	**2**	**3**	**4**
Expected return	10%	15%	18%	20%
Risk	20%	30%	25%	35%

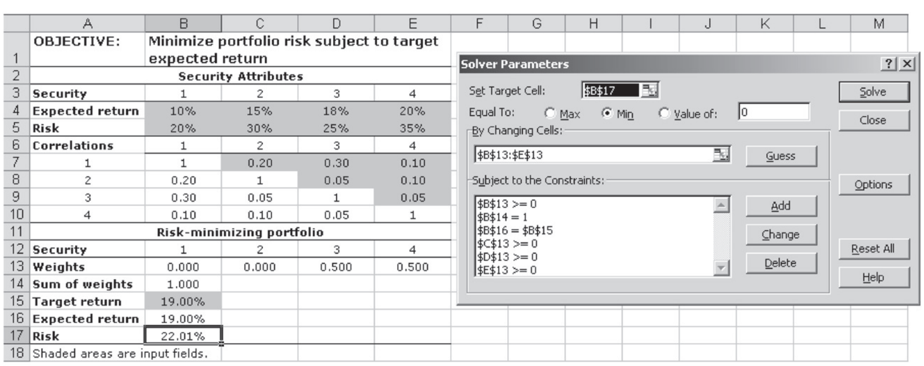
	Security Attributes			
Correlations	**1**	**2**	**3**	**4**
1	1	0.20	0.30	0.10
2	0.20	1	0.05	0.10
3	0.30	0.05	1	0.05
4	0.10	0.10	0.05	1

Solving this problem analytically is possible but cumbersome. Consequently, you may want to rely on Solver. The expressions for the expected return and risk of the four-asset portfolio are given by (3.3) and (3.4). These are programmed into cells B16 and B17 of the Excel file, **Four-asset portfolio.xls**. Verify that the computations are consistent with the formulas. To find the risk-minimizing portfolio with a target expected return of 19%, use Solver. The optimal allocations among the four risky securities are:



Note that the problem setup is identical to the two-asset case—we minimize portfolio risk subject to the constraints that the portfolio has a particular expected return and that the portfolio weights sum to one. The risk-minimizing portfolio has a risk level of 19.92%. The security weights in the risk-minimizing portfolio are –0.136, 0.243, 0.575, and 0.319 for securities 1 through 4, respectively. The negative weight on security 1 implies that it is sold short. The proceeds from the short sale, together with initial wealth, are invested in securities 2 through 4. The sum across all weights equals 1.

In many real-world portfolio allocation decisions, short sales of particular securities are not possible. To find the risk-minimizing portfolio with no short sales permitted, we impose four additional constraints in the risk-minimization problem. The Solver instructions are shown below. The risk-minimizing portfolio now consists of only two securities—50% in security 3 and 50% in security 4. No money is invested in securities 1 and 2. Note that the portfolio risk level is now 22.01%, well above the 19.92% when short sales were allowed. This stands to reason. The more the portfolio allocation decision is constrained, the less effective it becomes at reducing risk.



Portfolio Allocation with *n* Risky Securities and a Risk-Free Security

Tobin (1958) extended the Markowitz framework by introducing a risk-free security. To keep things simple at the outset, consider what happens when an individual created a two-security portfolio where one of the two securities is the risk-free security. The expected return on this portfolio is

$$E_P = X_1E_1 + (1 - X_1)r = r + (E_1 - r)X_1 \tag{3.7}$$

and the standard deviation of the portfolio return is

$$\sigma_P = \sqrt{X_1^2\sigma_1^2 + 2X_1(1 - X_1)\rho_{12}\sigma_1\sigma_2 + (1 - X_1)^2\sigma_2^2} = X_1\sigma_1 \tag{3.8}$$

Isolating X_1 in (3.8) and substituting into (3.7), we can generate any portfolio along the line

$$E_P = r + \left(\frac{E_1 - r}{\sigma_1}\right)\sigma_P \tag{3.9}$$

But since the individual can combine the risk-free security with any risky security or any risky security portfolio *S*, he will choose a portfolio that maximizes the slope of the line emanating from the risk-free rate in Figure 3.7, that is, he will borrow or lend with the risky security portfolio *S* that is tangent to the Markowitz efficiency frontier. Note that all other portfolios below the line have lower expected returns for a given level of risk. This line,

$$E_P = r + \left(\frac{E_S - r}{\sigma_S}\right)\sigma_P \tag{3.10}$$

now represents the individual's set of efficient portfolios. The individual's optimal portfolio is identified by mapping the individual's indifference curves on Figure 3.7. If his highest indifference curve is tangent to the left of *S*, his final portfolio will be a *lending* portfolio—some wealth invested in risky security portfolio *S* and some in the risk-free security. If it is tangent to the right of *S*, the individual not only has all of his wealth in *S*, but has also borrows additional funds, which are also invested in *S*.

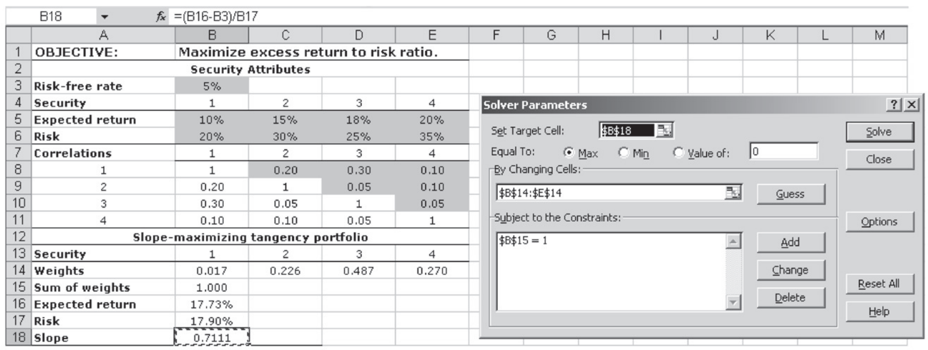
ILLUSTRATION 3.5 Identify composition of tangency risky-security portfolio.

Using the problem information from Illustration 3.4, find the composition of risky asset tangency portfolio. Then find the compositions of (a) the risk-minimizing portfolio with a target return of 24%, and (b) the expected return-maximizing portfolio with a risk tolerance of 15%.

The risky asset tangency portfolio is denoted *S* in Figure 3.7 and consists only of risky assets. To identify its composition, you must modify the objective function from Illustration 3.4. In place of minimizing portfolio risk subject to a given level of expected return, you maximize the ratio of the expected excess portfolio return to the standard deviation of portfolio return, that is,

$$\max\left(\frac{E_S - r}{\sigma_S}\right)$$

The necessary modifications are shown in the following illustration. The tangency portfolio has an expected return equal to 17.73% and a standard deviation of return equal to 17.90%. The weights invested in each of the risky assets are 0.017, 0.226, 0.487, and 0.270, and sum to one.


B18 =(B16-B3)/B17

	A	B	C	D	E
1	OBJECTIVE:	Maximize excess return to risk ratio.			
2		Security Attributes			
3	Risk-free rate	5%			
4	Security	1	2	3	4
5	Expected return	10%	15%	18%	20%
6	Risk	20%	30%	25%	35%
7	Correlations	1	2	3	4
8	1	1	0.20	0.30	0.10
9	2	0.20	1	0.05	0.10
10	3	0.30	0.05	1	0.05
11	4	0.10	0.10	0.05	1
12		Slope-maximizing tangency portfolio			
13	Security	1	2	3	4
14	Weights	0.017	0.226	0.487	0.270
15	Sum of weights	1.000			
16	Expected return	17.73%			
17	Risk	17.90%			
18	Slope	0.7111			

The next step in the portfolio allocation decision is compute the allocation between the risk-free asset and the risky asset portfolio, S. You can achieve this by specifying a target expected rate of return or a risk tolerance level. For a target expected return of 24%, for example, equation (3.7) says that

$$X_S = \frac{E_P - r}{E_S - r} = \frac{0.24 - 0.05}{0.1773 - 0.05} = 1.493$$

of your wealth should be allocated to risky portfolio S. In order to do this, of course, you must borrow 49.3% of your wealth at the risk-free rate of interest. *Risk tolerance* is usually defined as the maximum risk that you are willing to undertake and is specified as a standard deviation of return. For a risk tolerance of 15%, equation (3.8) says that

$$X_S = \frac{\sigma_P}{\sigma_S} = \frac{0.19}{0.1790} = 0.838$$

of your wealth should be allocated to risky portfolio S and 1 – 0.838 = 0.162 to risk-free bonds.

The figure that follows summarizes the results. All portfolios that lie on the straight line emanating from the risk-free rate of interest are efficient (i.e., maximize expected return for a given level of risk and/or minimize risk for a given level of expected return. All points on the line to the left of the point of tangency are *lending portfolios* in the sense that a positive amount is invested in risk-free bonds and a positive amount in risky portfolio S. The dot on the line immediately to the left of the tangency portfolio is the portfolio with a risk tolerance of 15%. All points on the line to the right of the tangency portfolio are *borrowing portfolios* in the sense that all initial wealth together with some risk-free borrowings is invested in the tangency portfolio S. The dot immediately to the right is the efficient portfolio with a target return of 24%.

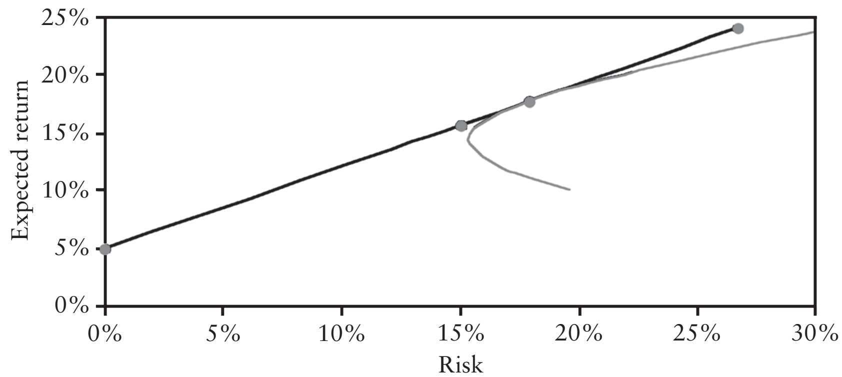
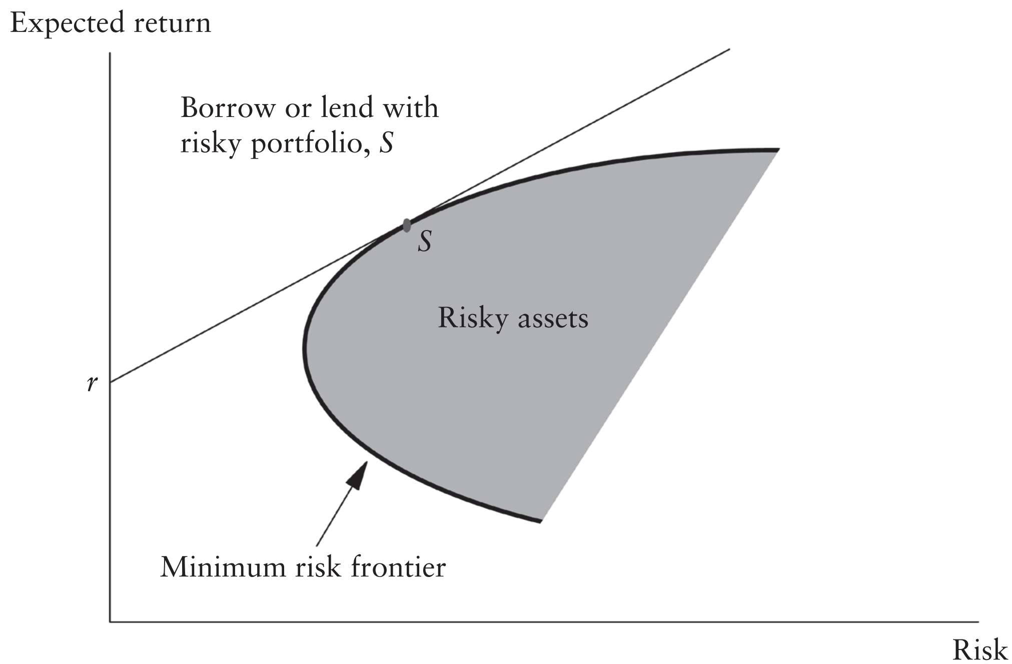


FIGURE 3.7 Minimum variance frontier with risk-free borrowing and lending.



CAPITAL ASSET PRICING MODEL

The capital asset pricing model (CAPM) specifies the formal relation between the expected rate of a security and its risk. Sharpe (1964) and Lintner (1965) independently developed this model by imposing two final assumptions on the Markowitz (1952)/Tobin (1958) framework,[6] that is, individuals share beliefs about the expected returns, standard deviations (variances), and correlations (covariances) of security returns, and individuals can all borrow and lend at the

[6] Jack L. Treynor independently developed the CAPM in a working paper dated 1962. Unfortunately, his paper was never published.

same risk-free rate of interest. The CAPM has three main results: (1) the capital market line, (2) the composition of the market portfolio, and (3) the security market line. We develop each in turn.

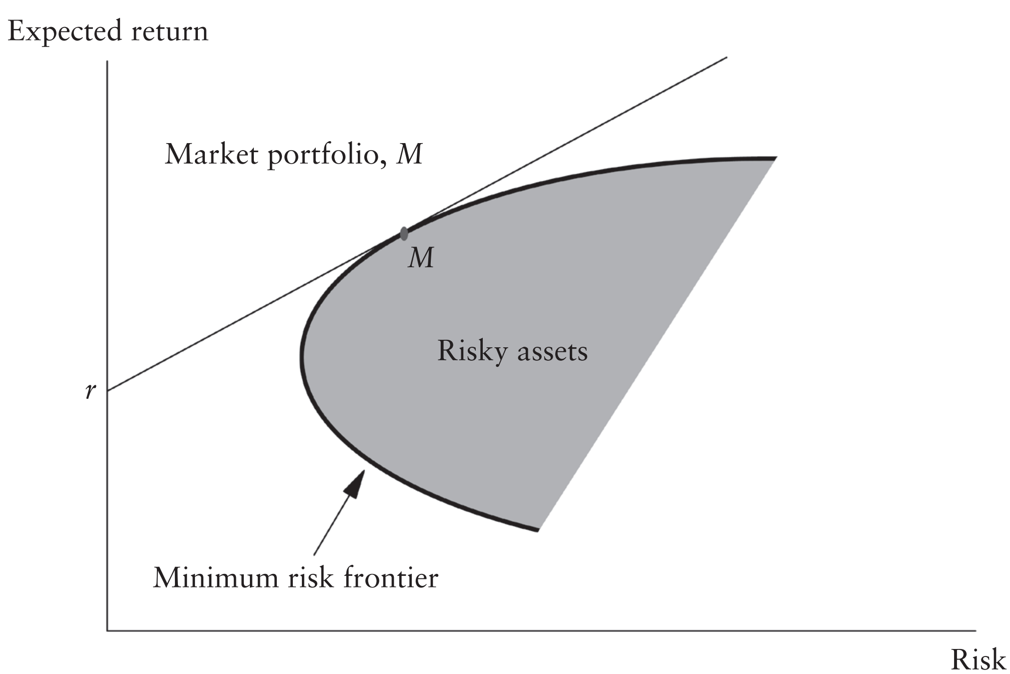
Capital Market Line

The *capital market line* represents the relation between expected return and risk for efficient portfolios. With common expectations regarding the expected returns, standard deviations, and correlations for risky assets and with all individuals being able to borrow and lend at the same risk-free interest rate, all individuals have the same tangency portfolio S in the Tobin framework. Since S must contain all risky assets in the economy, we relabel it M and call it the *market portfolio*. According to the model, all individuals will hold the market portfolio in combination with the risk-free asset. The relation between expected return and risk for these efficient portfolios is

$$E_P = r + \left(\frac{E_M - r}{\sigma_M}\right)\sigma_P \tag{3.11}$$

and is illustrated in Figure 3.8. An individual's allocation between the market portfolio and the risk-free asset will depend on his degree of risk aversion. A risk minimizer will invest all of his wealth in the risk-free asset. A risk-averse

FIGURE 3.8 Minimum variance frontier with risk-free borrowing and lending and common expectations.



individual will choose a portfolio that lies along the *capital market line* (3.11). If his highest indifference curve is tangent to the left of M, the final portfolio will be a *lending* portfolio—some wealth invested in M and some in the risk-free asset. If it is tangent to the right of M, the individual not only has all of his wealth in M, but has also borrows additional funds to invest in M.

Composition of the Market Portfolio

The allocation among risky securities in creating M can be established in a fairly intuitive fashion. We know that, if the market is in equilibrium, the total demand by individuals for risky asset i must equal total supply of asset i, that is,

$$\sum_{k=1}^{m} X_i^k w^k = V_i \tag{3.12}$$

where k represents the kth individual, m represents the number of individuals in the market, X_i^k is the proportion of k's risky asset wealth, w^k, invested in asset i, and V_i is the market value of asset i. We also know that, in equilibrium, total demand by individuals for risky assets must equal total supply, that is,

$$\sum_{k=1}^{m} w^k = \sum_{i=1}^{n} \sum_{k=1}^{m} X_i^k w^k = \sum_{i=1}^{n} V_i = V_M \tag{3.13}$$

where n is the number of risky assets in the market, and V_M is the market value of all risky assets. Finally, we know from the work of Markowitz that the allocation among risky securities in creating M (or any portfolio along the Markowitz efficiency frontier) is unique, that is, all investors allocate their risky asset wealth in the same proportions, that is, $X_i^k = X_i$ for all k. Thus, from (3.12) and (3.13), we find that

$$X_i = \frac{V_i}{V_M} \tag{3.14}$$

for all risky assets. In other words, the optimal proportion of risky asset wealth invested in risky asset i equals the market value of risky asset i divided by the market value of all risky assets.

Security Market Line

The security market line represents the equilibrium expected return/risk relation for risky assets. To identify this relation, we must identify asset i's contribution to the expected excess return and risk of the market portfolio. The market portfolio expected excess return and risk are defined as

$$E_M - r = \sum_{i=1}^{n} X_i(E_i - r) \tag{3.15}$$

and

$$\sigma_M^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \sigma_{ij} \tag{3.16}$$

The marginal contribution of asset i to the expected excess return of the market portfolio is

$$\frac{\partial E_M - r}{\partial X_i} = E_i - r \tag{3.17}$$

and the marginal contribution of asset i to the risk of the market portfolio is

$$\frac{\partial \sigma_M^2}{\partial X_i} = \sum_{j=1}^{n} X_j \sigma_{ij} = \sigma_{iM} \tag{3.18}$$

In equilibrium, all risky assets must have the same expected excess return/risk tradeoff, therefore

$$\frac{E_i - r}{\sigma_{iM}} = \frac{E_M - r}{\sigma_M^2} \tag{3.19}$$

The term on the right-hand side of (3.19) is often referred to as the *market price of risk*.[7] Rearranging,

$$E_i = r + (E_M - r)\frac{\sigma_{iM}}{\sigma_M^2} = R + (E_M - r)\beta_i \tag{3.20}$$

Equation (3.20) is called the *security market line* (SML). The SML represents the equilibrium expected return/risk relation for all risky securities in the marketplace. This includes stocks, bonds, currencies, and commodities, as well as, we shall see later, all types of derivative contracts. The three key results of the CAPM are summarized in Table 3.1.

[7] In a later chapter, we will rely on the concept of the market price of risk when we value options whose underlying asset is not actively traded.

TABLE 3.1 Summary of three key results from the Sharpe (1964)/Lintner (1965) capital asset pricing model.

Capital market line (CML):

The CML is the relation between expected return and risk for efficient portfolios, that is, portfolios with the highest level of expected return for a given level of risk or the lowest level of risk for a given level of expected return.

$$E_P = r + \left(\frac{E_M - r}{\sigma_M}\right)\sigma_P \tag{3.11}$$

where r is the risk-free rate of return, E_P and E_M are the expected returns on efficient portfolio P and the market portfolio M, respectively, and σ_P and σ_M are the standard deviation of the rate of return of efficient portfolio P and the market portfolio M, respectively.

Composition of market portfolio:

$$X_i = \frac{V_i}{V_M} \text{ for all } i \tag{3.14}$$

where X_i is the proportion of risky security wealth invested in the market portfolio, and V_i and V_M are the market values of risky security i and the market portfolio, respectively.

Security market line (SML):

The SML is the relation between expected return and risk for all risky securities in the market.

$$E_i = r + (E_M - r)\beta_i \tag{3.20}$$

where r is the risk-free rate of return, E_i and E_M are the expected returns of risky security i and the market portfolio M, respectively, and β_i is the relative systematic risk (i.e., "beta") of risky security i.

ILLUSTRATION 3.6 Estimate total risk and relative systematic risk of individual stock.

Estimate the total risk and relative systematic risk of IBM's stock using monthly stock returns. The historical data are provided in the Excel file, ***IBM.xls****. The file contains 60 months of returns for IBM, a value-weighted stock market index (i.e., a market portfolio proxy), and one-month Eurodollar time deposits (i.e., the money market rate used as a proxy for the risk-free rate of interest) over the past 60 months. Estimate the total risk and relative systematic risk of the excess returns of IBM and the market portfolio over the period January 2000 through December 2004.*

To estimate total risk, we simply compute the standard deviations of the different return series. This can be accomplished using the AVERAGE() and STDEV() functions in Microsoft Excel as shown below. The total risk of IBM over the estimation period is 10.38%, compared with 4.92% for the market portfolio. These values are usually reported on an annualized basis, which means we must multiply each by $\sqrt{12}$. On an annualized basis, the values are 35.95% and 17.06%, respectively. Note that the standard deviations of the excess returns (i.e., monthly return less the money market rate) are approximately the same at 10.37% and 4.95%, respectively. This result is expected since the standard deviation (i.e., total risk) of the money market rate is near 0.

B67 =STDEV(B4:B63)

	A	B	C	D	E	F
1	Monthly holding period returns (2000-2004)					
2		Returns		Risk-free	Excess returns	
3	Month	IBM	VW index	return	IBM	VW index
4	20000131	0.0406	-0.0398	0.0050	0.0356	-0.0448
5	20000229	-0.0836	0.0318	0.0047	-0.0883	0.0271
6	20000331	0.1484	0.0535	0.0051	0.1433	0.0484
61	20041029	0.0468	0.0178	0.0014	0.0453	0.0164
62	20041130	0.0520	0.0483	0.0018	0.0503	0.0465
63	20041231	0.0461	0.0352	0.0020	0.0441	0.0332
64						
65		Summary statistics				
66	Mean	0.0041	0.0002	0.0025	0.0016	-0.0023
67	StDev	0.1038	0.0492	0.0018	0.1037	0.0495
68						
69		OLS regression results				
70		alpha	t(alpha)	beta	t(beta)	R-squared
71	Returns	0.0038	0.3881	1.4421	7.1439	0.4681
72	Excess returns	0.0049	0.4950	1.4279	7.1025	0.4652

To estimate beta, we perform the regression,

$$R_{IBM,\,t} = \alpha + \beta R_{Market,\,t} + \varepsilon_t$$

where $R_{IBM,t}$ and $R_{Market,t}$ are the daily returns for IBM and the market index. (Recall a review of ordinary least squares (OLS) regression is provided in Appendix B of the book.) To do so, we can use the OPTVAL function,

OV_STAT_OLS_SIMPLE(*y, x, intercept, out*)

where *y* is dependent variable (i.e., the vector of IBM returns), *x* is the independent variable, *intercept* is an indicator variable set equal to "Y" or "y" if the regression includes an intercept and "N" or "n" if the regression does not include an intercept, and *out* is an indicator variable set equal to "H" or "h" if the output array is to be returns horizontally and "V" or "v" if the output array is to be returned vertically. To use the function, we need to highlight five adjacent cells either horizontally or vertically, insert the function, fill in the menu information, and then press the <Shift><Ctrl><Enter> keys simultaneously. The output array contains five elements: the estimated intercept term, its *t*-ratio for the null hypothesis that the intercept is 0, the estimated slope term (i.e., the "beta"), its *t*-ratio for the null hypothesis that the slope is 0, and the *R*-squared of the regression. As shown below, the estimated beta is 1.4421. The *R*-squared is 46.81%, which means that about 46.81% of IBM's total risk is market-related and that about 53.19% is diversifiable.

The second line of the regression results corresponds to the regression using excess returns, that is,

$$R_{IBM,\,t} - R_{MM,\,t} = \alpha + \beta(R_{Market,\,t} - R_{MM,\,t}) + \varepsilon_t$$

The estimated beta coefficient is virtually identical to the returns regression. This second regression is preferred since, according to the security market line of the CAPM, the expected value of the intercept term equals 0. Thus, an intercept term significantly different from 0 implies that the stock performed significantly better (worse) the expected according to the CAPM. For the excess return regression the estimated intercept is 0.0049 and its *t*-ratio is 0.4950. That means that IBM performed better than expected (0.0049 is positive), but that it is not significantly different than what was expected.

B71 f_x {=OV_STAT_OLS_SIMPLE(B4:B63,C4:C63,"Y","H")}

	A	B	C	D	E	F
1	Monthly holding period returns (2000-2004)					
2		Returns		Risk-free	Excess returns	
3	Month	IBM	VW index	return	IBM	VW index
4	20000131	0.0406	-0.0398	0.0050	0.0356	-0.0448
5	20000229	-0.0836	0.0318	0.0047	-0.0883	0.0271
6	20000331	0.1484	0.0535	0.0051	0.1433	0.0484
61	20041029	0.0468	0.0178	0.0014	0.0453	0.0164
62	20041130	0.0520	0.0483	0.0018	0.0503	0.0465
63	20041231	0.0461	0.0352	0.0020	0.0441	0.0332
64						
65		Summary statistics				
66	Mean	0.0041	0.0002	0.0025	0.0016	-0.0023
67	StDev	0.1038	0.0492	0.0018	0.1037	0.0495
68						
69		OLS regression results				
70		alpha	t(alpha)	beta	t(beta)	R-squared
71	Returns	0.0038	0.3881	1.4421	7.1439	0.4681
72	Excess returns	0.0049	0.4950	1.4279	7.1025	0.4652

PORTFOLIO PERFORMANCE MEASUREMENT

One of the many useful applications of the CAPM is portfolio performance measurement, that is, evaluating the historical performance of security portfolios on a risk-adjusted basis. Sometimes this is done in the context of assessing the performance of a fund manager. At other times, the measures are used to choose among available funds. Whatever the application, the most commonly-applied measures are shown in Table 3.2 and include the Sharpe (1966) ratio, the Modigliani and Modigliani (1997) *M*-squared, the Treynor (1965) ratio, and the Jensen (1968) alpha. In this section, we explain each in turn, showing that each assumes that the market behaves as it should with the CAPM. We also introduce a risk measure called semistandard deviation.

Total Risk Performance Measures

Two of the four performance measures—the Sharpe ratio and the M^2—are based on the total risk of the portfolio being evaluated. To understand the linkage between these measures and the CAPM, recall that, under the assumptions of the CAPM, all individuals hold efficient portfolios (i.e., portfolios that have the highest expected return for a given level of total risk). Recall also that all efficient portfolios lie along the capital market line (3.11).

The formula for computing the Sharpe ratio is given in Table 3.2. The notation on the right-hand side of the Sharpe ratio is different from the CAPM because to measure performance we rely on realized (ex post) returns rather than expected (ex ante) returns. Typically, monthly realized returns over the performance evaluation period are used. The length of the evaluation period ranges from as little as two years to more than a decade. It depends upon the objective. The parameters in the formulas are estimated from historical returns over the

TABLE 3.2 **Summary of CAPM-based portfolio performance measures.** $\bar{R}_f$, $\bar{R}_M$, and $\bar{R}_P$ are the mean returns of a "risk-free" money market instrument, the market, and the portfolio under consideration over the evaluation period, $\hat{\sigma}_M$ and $\hat{\sigma}_P$ are the standard deviations of the returns ("total risk") of the market and the portfolio, and $\hat{\beta}_P$ is the portfolio's systematic risk ("beta").

Total risk-based measures

$$\text{Sharpe ratio} = \frac{\bar{R}_P - \bar{R}_f}{\hat{\sigma}_P}$$

$$M^2 = (\bar{R}_P - \bar{R}_f)\left(\frac{\hat{\sigma}_M}{\hat{\sigma}_P}\right) - (\bar{R}_M - \bar{R}_f)$$

Systematic risk based measures

$$\text{Treynor ratio} = \frac{\bar{R}_P - \bar{R}_f}{\hat{\beta}_P}$$

$$\text{Jensen's alpha} = \bar{R}_P - \bar{R}_f - \hat{\beta}_P(\bar{R}_M - \bar{R}_f)$$

evaluation period: $\bar{R}_f$, $\bar{R}_M$, and $\bar{R}_P$ are the mean monthly returns of a "risk-free" money market instrument, the market, and the portfolio under consideration over the evaluation period, and $\hat{\sigma}_M$ and $\hat{\sigma}_P$ are the standard deviations of the returns ("total risk") of the market and the portfolio. Now assume that the portfolio P behaved exactly as predicted over the evaluation period (i.e., expectations are realized). Under this assumption, the Sharpe ratio may be written

$$\text{Sharpe ratio} = \frac{\bar{R}_P - \bar{R}_f}{\hat{\sigma}_P} = \frac{E_P - r}{\sigma_p} \tag{3.21}$$

But according to the CML (3.11), the expected excess return per unit of total risk for efficient portfolio P equals the expected excess return per unit of total risk for the market portfolio M, and, with expectations being realized,

$$\text{Sharpe ratio} = \frac{\bar{R}_P - \bar{R}_f}{\hat{\sigma}_P} = \frac{E_P - r}{\sigma_p} = \frac{E_M - r}{\sigma_M} = \frac{\bar{R}_M - \bar{R}_f}{\hat{\sigma}_M} \tag{3.22}$$

In other words, the benchmark realized return/total risk ratio is that of the market portfolio, that is, the rightmost term in (3.22). If portfolio being evaluated performed as expected under the CAPM, its realized return/total risk ratio should be the same as the market portfolio,

$$\frac{\overline{R}_P - \overline{R}_f}{\hat{\sigma}_P} = \frac{\overline{R}_M - \overline{R}_f}{\hat{\sigma}_M} \tag{3.23}$$

If, on the other hand, we were able to identify underpriced securities in the selection of our portfolio and our portfolio outperformed the market, the portfolio's Sharpe ratio would exceed that of the market.

The M^2 measure of performance levers portfolio P in such a way that its total risk equals that of the market portfolio, and then examines the difference between the excess returns of the portfolio and the market. The term, $\hat{\sigma}_M / \hat{\sigma}_P$, in the M^2 formula shown in Table 3.2 is the degree of leverage. If the portfolio's total risk was below (above) the market's during the evaluation period, the ratio will exceed (be below) one and the excess return of the portfolios will be levered up (down) in order to match the total risk of the market, that is,

$$\hat{\sigma}_P(\hat{\sigma}_M / \hat{\sigma}_P) = \hat{\sigma}_M$$

With equal risk levels, we can compare the levered portfolio's return with the market return directly. Assuming the portfolio behaved as expected under the CAPM and expectations were realized, the realized abnormal performance of the portfolio, as measured by M^2, is

$$M^2 = (\overline{R}_P - \overline{R}_f)\left(\frac{\hat{\sigma}_M}{\hat{\sigma}_P}\right) - (\overline{R}_M - \overline{R}_f) = (E_P - r)\left(\frac{\sigma_M}{\sigma_P}\right) - (E_M - r) = 0 \tag{3.24}$$

Where $M^2 > 0$, portfolio P outperformed the market on a risk-adjusted basis, and vice versa.

Systematic Risk Performance Measures

The remaining two performance measures—the Treynor ratio and the Jensen alpha—are based on systematic risk and are the counterparts to the Sharpe ratio and M^2, respectively. To understand the linkage between the systematic risk performance measures, recall that, under the assumptions of the CAPM, all risky securities lie along the security market line (3.20). Since a portfolio is nothing more than a weighted combination of securities, it is also the case that portfolios lie along the SML, that is,

$$E_P = r + (E_M - r)\beta_P \tag{3.25}$$

The formula for the Treynor ratio is given in Table 3.2. The portfolio's realized systematic risk or beta, $\hat{\beta}_P$ is estimated by an ordinary least squares, time-series regression of the excess returns of the portfolio on the excess returns of the market, that is,

$$R_{P,t} - R_{f,t} = \alpha + \beta_P(R_{M,t} - R_{f,t}) + \varepsilon_{P,t} \tag{3.26}$$

Like in the case of the Sharpe ratio, the realized excess return per unit of risk for the portfolio should, in equilibrium, be equal to the excess return of the market portfolio, however, since $\beta_M = 1$,

$$\text{Treynor ratio} = \frac{\overline{R}_P - \overline{R}_f}{\hat{\beta}_P} = \frac{E_P - r}{\beta_P} = \frac{E_M - r}{\beta_M} = E_M - r = \overline{R}_M - \overline{R}_f \tag{3.27}$$

If a portfolio outperformed the market on a risk-adjusted basis, its Treynor ratio will exceed the realized excess return of the market.

Jensen's alpha is essentially the intercept term in a regression of the excess returns of the portfolio on the excess returns of the market (3.26). If the market behaves according to the CAPM and expectation are realized,

$$\text{Jensen's alpha} = \overline{R}_P - \overline{R}_f - \hat{\beta}_P(\overline{R}_M - \overline{R}_f) = E_P - r - \beta_P(E_M - r) = 0 \tag{3.28}$$

If the estimated value of Jensen's alpha, $\hat{\alpha}_P$, is greater than zero, the portfolio outperformed the market on a risk-adjusted basis.

Alternative Risk Measures

The performance measures in Table 3.2 are occasionally criticized because the Sharpe (1964)/Lintner (1965) mean/variance capital asset pricing model assumes investors measure total portfolio risk by the standard deviation of returns. Among other things, this implies that investors view an unexpectedly large positive return with the same distaste as an unexpectedly large negative return. Common sense dictates otherwise. Investors are willing to pay for the chance of a large positive return (i.e., positive skewness) holding other factors constant, but will want to be paid for taking on negative skewness. Since the standard performance measures do not recognize these premiums/discounts, portfolios with positive skewness will appear to underperform the market on a risk-adjusted basis, and portfolios with negative skewness will appear to overperform.

Ironically, while the Sharpe/Linter CAPM is based on the mean/variance portfolio theory of Markowitz (1952), it was Markowitz (1959) who first noted that using standard deviation to measure risk is too conservative since it regards all extreme returns, positive or negative, as undesirable. Markowitz (1959, Ch. 9) advocates the use of semivariance or semistandard deviation as a total risk measure.[8] To understand the relation between standard deviation and semistandard deviation, begin with total risk as measured traditionally using the standard deviation of excess returns, that is,

[8] Indeed, in his Nobel Prize acceptance speech, the Markowitz (1991) continues to argue semivariance seems more plausible than variance as a measure of risk.

$$\text{Standard deviation}_i = \sqrt{\frac{\sum_{t=1}^{T}(R_{i,t} - R_{f,t} - k)^2}{T}} \tag{3.29}$$

where k is the mean realized excess return.[9] With no loss of generality, we can write (3.29) as

$$\text{Standard deviation}_i = \sqrt{\frac{\sum_{t=1}^{T}\min(R_{i,t} - R_{f,t} - k, 0)^2}{T} + \frac{\sum_{t=1}^{T}\max(R_{i,t} - R_{f,t} - k, 0)^2}{T}} \tag{3.30}$$

Under the square root sign, we now have two terms. The first is the sum of the squared deviations when the excess return is below k and the second is the sum of the squared deviations when the excess return is above k. Suppose we are willing to conjecture that individuals care about risk only to the extent that their risky asset portfolio does not produce a rate of return as high as the risk-free rate of return. To create such a risk measure, we set $k = 0$ and drop the second term under the square root sign. In the context of performance measurement, semistandard deviation can be defined as the square root of the average of the squared deviations from the risk-free rate of interest, where positive deviations are set equal to zero, that is,

$$\text{Semistandard deviation}_i = \sqrt{\frac{\sum_{t=1}^{T}\min(R_{i,t} - R_{f,t}, 0)^2}{T}} \tag{3.31}$$

where $i = M,P$. Returns on risky assets, when they exceed the risk-free rate of interest, do not affect risk. To account for possible asymmetry of the portfolio return distribution, we recompute the total risk portfolio performance measures (1) and (2) using the estimated semideviations of the returns of the market and the portfolio are inserted for $\hat{\sigma}_M$ and $\hat{\sigma}_P$.[10]

The systematic risk-based portfolio performance measures (3) and (4) also have theoretical counterparts in a semivariance framework. The only difference lies in the estimate of systematic risk. To estimate the beta, a time-series regression *through the origin* is performed using the excess return series of the market and the portfolio. Where excess returns are positive, they are replaced with a zero value. The time-series regression specification is

$$\min(R_{P,t} - R_{f,t}, 0) = \beta_p \min(R_{M,t} - R_{f,t}, 0) + \varepsilon_{P,t} \tag{3.32}$$

[9] Technically speaking, the denominator should be $T - 1$ since we use up a degree of freedom when we estimate the mean excess return.

[10] The ratio of realized excess return relative to the semistandard deviation of return is commonly referred to as the Sortino ratio. See Sortino and Van der Meer (1991).

ILLUSTRATION 3.7 Estimate performance of CBOE's buy-write index portfolio (BXM).

The Excel file, ***Performance measurement.xls****, contains monthly returns of the CBOE's BXM index, the S&P 500 index (i.e., the "market" index), and a 30-day money market instrument (i.e., 30-day Eurodollar time deposits) for the period January 1996 through December 2004. Based on the monthly returns, compute the performance of the BXM portfolio, and comment on the results.*

The usual way to compute the performance measures is to compute the mean and standard deviation (and semistandard deviation) of the monthly (excess) return series as well as each series "beta" and then substitute the estimated parameters in the formulas reported in Table 3.2.

The first step is to compute the means and standard deviations of the return variables. The second step is to estimate the beta of the BXM using OLS regression. The results are as follows:

B122 fx =E112/B113

	A	B	C	D	E	F
1	Month	Monthly returns			Excess returns	
2	end	BXM	SPX	MM	BXM	SPX
3	19960131	0.00846	0.03404	0.00513	0.00333	0.02891
4	19960229	-0.00229	0.00927	0.00428	-0.00657	0.00499
5	19960329	0.01481	0.00963	0.00418	0.01063	0.00545
108	20041029	0.00913	0.01528	0.00145	0.00768	0.01383
109	20041130	0.00260	0.04046	0.00175	0.00084	0.03871
110	20041231	0.02559	0.03403	0.00195	0.02364	0.03208
111						
112	Mean	0.00872	0.00873	0.00342	0.00530	0.00531
113	StDev	0.03311	0.04703	0.00170	0.03301	0.04693
114						
115		OLS regression results				
116		alpha	t(alpha)	beta	t(beta)	R-squared
117	Returns	0.00336	2.11009	0.61452	18.40195	0.76160
118	Excess returns	0.00204	1.29886	0.61343	18.35223	0.76062
119						
120						
121	Measure	BXM	SPX			
122	Sharpe ratio	0.16002	0.11293			
123	M-squared	0.221%				
124	Treynor ratio	0.00864	0.00531			
125	Jensen's alpha	0.204%				

The results are interesting in a number of respects. First, note that the mean monthly returns of the BXM and the S&P 500 are virtually identical—0.872% versus 0.873%—while the standard deviation of return for the BXM is about 2/3 of the S&P 500—3.311% versus 4.703%. This is the first indication that the BXM outperformed the market during the evaluation period. Second, note that the return regression produces virtually the same beta as the excess regression. This is, again, a reflection of the fact that the variance of the money market rate (i.e., the proxy for the risk-free rate) is small in relation to the variances of the BXM and the S&P 500 returns. Finally, all four performance measures indicate that the BXM outperformed the market on a risk-adjusted basis during the evaluation period, January 1996 through December 2004. The Sharpe ratio of the BXM is computed as the realized excess return on the BXM divided by its standard deviation. To evaluate performance, we must also compute the Sharpe ratio for the market portfolio. Since 0.16002 exceeds 0.11293, the BXM outperformed the market. The M^2 is 0.221%, which means that the BXM outperformed the market portfolio by 0.221% or 22 basis points per month. For this measure, as well as the Jensen alpha, no benchmark measure is necessary since they market performance is implicitly incorporated. The Treynor ratio as well as Jensen's alpha also show that the BXM produced a higher than expected return on a risk-adjusted basis.

Preprogrammed functions for the performance measures are also included in the OPTVAL Function Library. The syntax for the Sharpe Ratio function call, for example, is

OV_PERF_SR(*RetP, RetF, Measure*)

where *RetP* is the vector of portfolio returns whose performance is to be evaluated, *RetF* is the vector of money market rates, and *Measure* is an indicator variable whose value is 0 for total risk measured using the standard deviation of return and 1 for total risk measured as semistandard deviation. The panel below illustrates. The remaining function calls are:

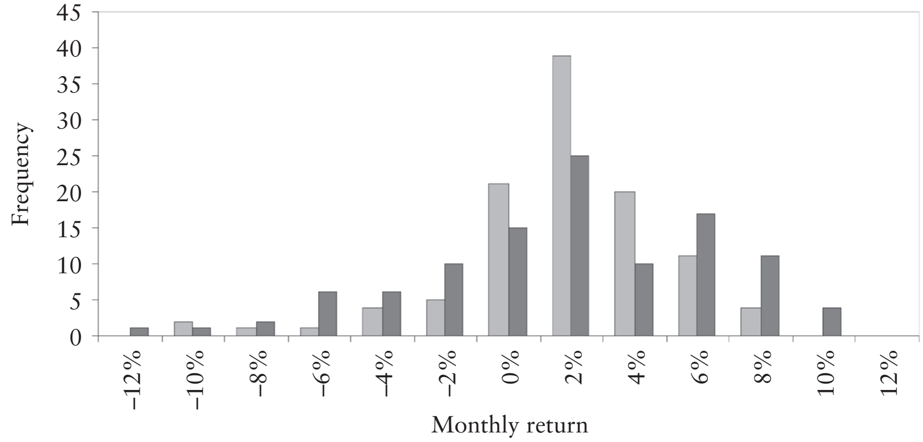
OV_PERF_M2(*RetP, RetM, RetF, Measure*)
OV_PERF_TR(*RetP ,RetM, RetF, Measure*)
OV_PERF_JA(*RetP, RetM, RetF, Measure*)

where *RetM* is the vector of market returns and the remaining terms are defined as before.

C123 =OV_PERF_SR($B$3:$B$110,$D$3:$D$110,1)

	A	B	C	D	E	F
1	Month	Monthly returns			Excess returns	
2	end	BXM	SPX	MM	BXM	SPX
3	19960131	0.00846	0.03404	0.00513	0.00333	0.02891
4	19960229	-0.00229	0.00927	0.00428	-0.00657	0.00499
5	19960329	0.01481	0.00963	0.00418	0.01063	0.00545
108	20041029	0.00913	0.01528	0.00145	0.00768	0.01383
109	20041130	0.00260	0.04046	0.00175	0.00084	0.03871
110	20041231	0.02559	0.03403	0.00195	0.02364	0.03208
111						
112	Mean	0.00872	0.00873	0.00342	0.00530	0.00531
113	StDev	0.03311	0.04703	0.00170	0.03301	0.04693
114						
115		OLS regression results				
116		alpha	t(alpha)	beta	t(beta)	R-squared
117	Returns	0.00336	2.11009	0.61452	18.40195	0.76160
118	Excess returns	0.00204	1.29886	0.61343	18.35223	0.76062
119						
120		BXM		SPX		
121		Mean-	Mean-	Mean-	Mean-	
122	Measure	stddev	semi-stddev	stddev	semi-stddev	
123	Sharpe ratio	0.16002	0.22218	0.11293	0.16281	
124	M-squared	0.221%	0.194%			
125	Treynor ratio	0.00864	0.00815	0.00531	0.00531	
126	Jensen's alpha	0.204%	0.185%			

The performance results for the BXM and S&P 500 portfolios computed using the appropriate OPTVAL functions are as shown above. The numerical values under the column heading "Mean-stddev" are exactly the same as those computed by hand. The values under the column heading "Mean-semi-stddev" also show that the BXM performed better than the market on a risk-adjusted basis, however, the performance is not as high under the mean/semistandard deviation framework as the CAPM's mean/standard deviation framework. The reason is simple. The returns of the BXM (the lighter shaded bars) are more negatively skewed than those of the market (the darker shaded bars). Since the semistandard deviation focuses only on returns below the risk-free interest rate, the negative skewness relative to the market results in lower performance measures. A histogram of the BXM and S&P 500 returns over the evaluation period are shown below. But even after extracting a penalty for negative skewness, the BXM strategy appears to outperform the market (as proxied by the S&P 500 index portfolio).



SUMMARY

Effective risk management requires a thorough understanding of the tradeoff between expected return and risk. This chapter reviews expected return/risk mechanics. A risky security is characterized by its expected return, standard deviations of return, and the correlations of its returns with all other securities in the marketplace. Given this characterization, we show how a risk-averse individual allocates his wealth among securities under a variety of scenarios and constraints. Individual security demands are then aggregated across all individuals in the marketplace to create a market portfolio. The marginal contribution of each security to the expected excess return and risk of the market portfolio identifies the equilibrium expected return/risk relation for risky securities. The expected return/risk relations, known collectively as the capital asset pricing model, will be used again and again throughout the chapters of this book, as it guides us in analyzing and designing risk management strategies. We apply the relations in this chapter to examine historical portfolio performance.

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