CHAPTER 9

Valuing Options Numerically

The last two chapters focused on valuing options analytically. Analytical valuation equations were possible because, in general, the options were European-style with only one exercise opportunity. For other types of options, the valuation problem is not so simple. With American-style options, for example, there are an infinite number of early exercise opportunities between now and the expiration date, and the decision to exercise early depends on a number of factors including all subsequent exercise opportunities. An analytical solution for the American-style option valuation problem (i.e., a valuation equation) has not been found.[1] The same is true for many Asian-style options (e.g., options written on an arithmetic average) and many European-style options with multiple sources of underlying price risk (e.g., spread options). In such cases, options must be valued numerically. Moreover, even in instances where analytical solutions to option contract values are possible (e.g., accrual options), numerical methods are often easier to apply.

The purpose of this chapter is to discuss numerical methods for valuing options. All of them are developed within the Black-Scholes/Merton (BSM) option valuation framework. The underlying asset's price is assumed to follow a geometric Brownian motion (i.e., to be log-normally distributed at any future instant in time), and a risk-free hedge between the option and its underlying asset(s) is possible. Three of the methods involve replacing the continuous Brownian diffusion with a process that involves discrete jumps. The *binomial method,* for example, assumes that the asset price moves to one of two levels over the next increment in time. The size of the move and its likelihood are chosen in a manner so as to be consistent with the log-normal asset price distribution. In a similar fashion, the *trinomial method,* described in the second section, allows the asset price to move to one of three levels over the next increment in time. The third section describes a *Monte Carlo simulation* technique, which uses a discretized version of geometric Brownian motion to enumerate every possible path that the asset's price may take over the life of the option. The *quadratic approximation method*, discussed in the fourth section, addresses the value of early exercise by modifying the BSM partial differential equation.[2] As important

[1] The exception is American-style call options on assets with zero or negative income rates.

as valuation, however, is risk measurement. The fifth section of the chapter describes how to compute the risk characteristics (i.e., the Greeks) of options using numerical methods. Finally, the sixth section contains a brief summary.

BINOMIAL METHOD

The binomial method is the most popular approximation for valuing American-style options. It is easy to implement and flexible enough to handle a wide range of option valuation problems. Under the binomial method, the option's life is divided into fixed-length time steps, and, in each time step, the asset price is allowed to jump up or down. Defining n as the number of time steps, each time increment has length $\Delta t = T/n$, where T is the time remaining to expiration of the option.

The binomial distribution is characterized by the size of its price steps and their probabilities. We must choose the parameters in such a way that the mean and the variance of the discrete binomial distribution are consistent with the mean and the variance of the continuous log-normal distribution underlying the BSM model. To make matters simple as possible, we will focus on the logarithm of the asset price at the end of the time increment Δt, which, under the BSM assumptions is normally distributed with mean $\ln S + \mu\Delta t$ and variance $\sigma^2\Delta t$. First, we set the mean of the binomial distribution equal to the mean of the logarithm of asset price distribution, that is,

$$p(\ln S + v) + (1-p)(\ln S + w) = \ln S + \mu\Delta t \tag{9.1}$$

In (9.1), p is the probability that the logarithm of asset price changes by v, and $1 - p$ is the probability that the logarithm of asset price changes by w. Note that we have made no assumption yet regarding the sizes of v and w, although we will do so shortly. The $\ln S$ terms fall out of (9.1), and we are left with the mean constraint,

$$pv + (1-p)w = \mu\Delta t \tag{9.2}$$

Next we set the variance of the binomial distribution equal to the variance of the logarithm of asset price distribution, that is,

$$p(\ln S + v - (\ln S + \mu\Delta t))^2 + (1-p)(\ln S + w - (\ln S + \mu\Delta t))^2 = \sigma^2\Delta t \tag{9.3}$$

The $\ln S$ terms are again irrelevant, and, with a little additional algebra, equation (9.3) becomes the variance constraint,

$$pv^2 + (1-p)w^2 = \sigma^2\Delta t + \mu^2\Delta t^2 \tag{9.4a}$$

Equation (9.4a) is a little unusual in the sense that it has a term that includes the time increment squared, Δt^2. In applying the binomial method to value options,

[2] Recall Appendix 7D in Chapter 7.

however, a large number of time steps is usually used, so Δt is very small and terms with higher order values of Δt can safely be ignored. Ignoring the higher order term, the variance constraint is

$$pv^2 + (1-p)w^2 = \sigma^2 \Delta t \tag{9.4b}$$

Note that the terms on the right-hand side of (9.2) and (9.4), μ and σ^2, are known parameters of the normal distribution of the logarithm of asset prices. Our objective is to find the values of v, w, and p that make the mean and variance of the binomial distribution consistent with the mean and the variance of the normal distribution of the logarithm of asset prices. With two equations (i.e., (9.2) and either (9.4a) or (9.4b)) and three unknowns, we cannot solve for the parameters v, w, and p uniquely, so another constraint must be imposed. Below, we discuss the constraints used in two well-known implementations of the binomial method.

Cox-Ross-Rubinstein (1979) Parameters

Cox, Ross and Rubinstein (1979) (hereafter CRR) impose the symmetry constraint, $w = -v$, where v is a positive increment. This implies that, over the next increment in time Δt, the asset price will either rise to level, $\ln S + v$, or fall to level, $\ln S - v$. CRR use (9.4b) to tie the variance of the binomial distribution to the variance of the logarithm of asset prices. The value of v that satisfies (9.4b) is

$$v = \sigma\sqrt{\Delta t} \tag{9.5}$$

With v and w known, we turn to finding the level of probability, p. Substituting (9.5) into the mean condition (9.2) and rearranging to isolate p, the probability of an up-step is

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{\Delta t} \tag{9.6}$$

As in Chapter 7, we adopt the practice of using the continuously compounded mean rate of price appreciation, α, rather than the mean continuously compounded rate, μ. Substituting $\mu = \alpha - 0.5\sigma^2$ into (9.6), we get

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{\alpha - 0.5\sigma^2}{\sigma}\right)\sqrt{\Delta t} \tag{9.7}$$

Also recall that under the BSM option valuation framework, a risk-free hedge can be formed between the option and its underlying asset. This implies that option valuation is not sensitive to the risk preferences of an individual, so in the interest of mathematical tractability, we assume risk-neutrality, in which case the continuously compounded mean rate of price appreciation, α, becomes the asset's cost of carry rate (i.e., $\alpha = b$) and the probability of an up-step is

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{b - 0.5\sigma^2}{\sigma}\right)\sqrt{\Delta t} \tag{9.8}$$

Jarrow-Rudd (1983) Parameters

In another well-known implementation of the binomial method, Jarrow and Rudd (1983) (hereafter JR) impose the constraint that the up-step and down-step probabilities are both equal to $p = 1/2$. This means that the constraint that matches the mean of the binomial distribution with the mean of the change in the logarithm of prices (9.2) may be written

$$v + w = 2\mu\Delta t \tag{9.9}$$

To express the variance constraint, JR use (9.4a). With $p = 1/2$, the variance constraint can be rewritten as

$$v^2 + w^2 = 2\sigma^2\Delta t + \frac{1}{2}(4\mu^2\Delta t^2) \tag{9.10}$$

Substituting the square of (9.9) into the parentheses on the right-hand side of (9.10), rearranging, factoring, taking the square root and then simplifying, we get

$$v - w = 2\sigma\sqrt{\Delta t} \tag{9.11}$$

Equations (9.9) and (9.11) can now be used to identify u and v. With the probability set equal to 1/2, the up-step coefficient is

$$v = \mu\Delta t + \sigma\sqrt{\Delta t} = (b - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t} \tag{9.12a}$$

and the down-step coefficient is

$$w = \mu\Delta t - \sigma\sqrt{\Delta t} = (b - 0.5\sigma^2)\Delta t - \sigma\sqrt{\Delta t} \tag{9.12b}$$

In going from the middle term to the last term in (9.12a) and (9.12b), we are, of course, invoking an assumption of risk-neutrality.

Applying the Binomial Method

With the binomial distribution parameters now defined, we turn to applying the model. Applying the binomial method has three steps. To illustrate each step, we value a two-year American-style put option with an exercise price of 55. The'umed to be a foreign currency (FX rate) whose current price is 50 and whose volatility rate is 20%. The domestic rate of interest is assumed to be 5%, and the foreign rate of interest, 2%. The expected risk-neutral rate of price appreciation of the currency is therefore 3%. We apply the binomial method first using the CRR parameters and then, in the interest of comparison, using the JR parameters.

Step 1: Create the Asset Price Lattice The first step in the binomial method is to use the computed parameters of the binomial process to trace out every conceivable path that the asset price may take between now and the option's expiration. Thus far we have focused on movements in the logarithm of asset price. To generate a lattice for $\ln S$, we must set the number of time steps, n. The greater the number of time steps, the higher the accuracy of the approximation, but the higher the computational cost. The time increment is $\Delta t = T/n$, where T is the time to expiration of the option. We set the number of time steps in our illustration to 2. The time increment Δt is therefore one year.

Figure 9.1 shows the possible paths that the logarithm of asset price may take during the option's life. Four paths are possible: (1) up, up, (2) up, down, (3) down, up, and (4) down, down. Note that if the asset price goes up in the first year and down in the second (or vice versa), the logarithm of asset price is back where it started. This is the symmetry condition imposed by CRR. The size of the jump in the logarithm of asset price from period to period, v, is given by (9.5). From the problem information, we can compute the price increment, $v = 0.20\sqrt{1} = 0.20$. The logarithm of the current asset price is ln50 = 3.912. Applying the price increment to identify the values of the nodes of the lattice in year 1 and year 2, we can trace out all of the possible movements of the logarithm of asset prices over the life of the option. These movements are shown in Figure 9.2.

FIGURE 9.1 Two-period lattice showing the logarithm of asset price at different times during the option's life.

Year	0	1	2
			$\ln S + 2v$
		$\ln S + v$	
	$\ln S$		$\ln S$
		$\ln S - v$	
			$\ln S - 2v$

FIGURE 9.2 Two-period lattice showing numerical values for the logarithm of asset price at different times during the option's life.

Time	0	1	2
			4.312
		4.112	
	3.912		3.912
		3.712	
			3.512

FIGURE 9.3 Two-period lattice showing the asset price at different times during the option's life.

Year	0	1	2
			uuS
		uS	
	S		S
		dS	
			ddS

In general, individuals who apply the binomial method prefer to see the lattice expressed in asset price rather than the logarithm of asset price. To create such a lattice, we can raise the logarithm of the asset price shown in Figure 9.2 to the power of e. Alternatively, we can redefine the problem from one which uses absolute price changes to one which uses relative price changes. To do so, recognize that an additive jump of v in the logarithm of asset price S, (i.e., $\ln S + v$), is equivalent to a multiplicative jump of u in the asset price S, (i.e., Su), where

$$u = e^{v} = e^{\sigma\sqrt{\Delta t}}$$

and an additive jump of $-v$ in the logarithm of asset price S, (i.e., $\ln S - v$), is equivalent to a multiplicative jump of d in the asset price S (i.e., Sd), where $d = e^{-v} = 1/u$. Like before, successive up and down steps return the asset to its original price, as shown in Figure 9.3. At the end of one year, the possible asset prices are uS and dS. At the end of year 2, the possibilities are that the asset price moves from uS to uuS or S and from dS to S or ddS. In this two-period problem, all possible paths that the asset price may follow between and the option's expiration are shown in Figure 9.3. Since the volatility rate is 20% and the time increment is one year, the values of u and d are

$$u = e^{0.20\sqrt{1}} = 1.2214 \quad \text{and} \quad d = \frac{1}{u} = \frac{1}{1.2214} = 0.8187$$

Applying these coefficients to the current asset price of 50 produces Figure 9.4, and computing the numerical values in each cell of Figure 9.4 produces Figure 9.5. Note that the node values in Figure 9.5 are simply the values in Figure 9.2 raised to the power of e.

Step 2: Value Option at Expiration With all of the asset price nodes at the option's expiration computed, we turn to valuing the option. At this stage of the numerical procedure, the only option values that we can compute are those on the expiration date. The value of the option at expiration equals the maximum of its exercisable value and 0. The terminal values of the put in our illustration are shown in boldface in Figure 9.6.

FIGURE 9.4 Two-period lattice showing numerical values for the asset price at different times during the option's life.

Time	0	1	2
			$1.2214^2(50)$
		1.2214(50)	
	50		1.2214(0.8187)50
		0.8187(50)	
			$0.8187^2(50)$

FIGURE 9.5 Two-period lattice showing numerical values for the asset price using the Cox-Ross-Rubinstein parameters.

Time	0	1	2
			74.59
		61.07	
	50		50.00
		40.94	
			33.52

FIGURE 9.6 Two-period lattice showing the terminal values of put option written on asset using the Cox-Ross-Rubinstein parameters.

Time	0	1		2
			max(0,55 – 74.59) =	**0.00** 74.59
		61.07		
			max(0,55 – 50) =	**5.00** 50.00
	50			
		40.94		
			max(0,55 – 33.52) =	**21.48** 33.52

Step 3: Value Option at Earlier Nodes by Taking the Present Value of the Expected Future Value The next step is to value the option at earlier nodes. This is done recursively move one step back in time and valuing the option at each vertical node by taking the present value of the expected future value of the option based on the two nodes lying immediately to its right. This means that, in order to identify the value of the option at the upper node in year 1 of Figure 9.6, we need to know the probability that the asset price will move from 61.07 to 74.59 and the probability that the asset price will move from 61.07 to 50.

The expression used for computing the probability of an up-step in the CRR procedure is (9.8). Evaluating the probability,[3] we get

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{0.03 - 0.5(0.20^2)}{0.20}\right)\sqrt{1} = 0.525$$

The probability of a down-step in asset price is the complement of the probability of an up-step, that is, 1 – 0.525 = 0.475.

With the probabilities in hand, we can compute the expected future value of the put conditional upon being in year 1 with an asset price level 61.09. The *expected future value* (EFV) is

$$EFV = 0.525(0) + 0.475(5) = 2.375$$

Under risk-neutrality, the *present value of the expected future value* (PVEFV) is

$$PVEFV = e^{-r\Delta t}(2.375) = e^{-0.05(1)}(2.375) = 2.26$$

Thus we have identified the value of the put in year 1 conditional upon the asset price being 61.07, as shown in Figure 9.9. The year 1 value of the put conditional upon the asset price being 40.94 is obtained using the same procedure and equals 12.20. Taking the present value of the expected future value in year 0 reveals that the current value of the put is 6.64.

The computations supporting the option values shown in Figure 9.7 indicate that the current value of the put is 6.64. But what is the style of the put option we have valued? The answer is European-style. In computing the current value, we did not account for the prospect of early exercise. In applying the binomial method to value American-style options, we must check whether the put should have been exercised early at the beginning and any of the intermediate nodes of the lattice.

FIGURE 9.7 Current and intermediate values of European-style put option written on asset using the Cox-Ross-Rubinstein parameters.

Time	0	1	2
			0.00
			74.59
		2.26	
		61.07	
	6.64		**5.00**
	50.00		50.00
		12.20	
		40.94	
			21.48
			33.52

[3] Occasionally, the probability for the CRR method is approximated using $p = (e^{b\Delta t} - t)/(u - d)$. We refer to this practice as the *simple method*.

Step 3a: Check for Optimal Early Exercise In our two-period illustration, the only opportunities for early exercise occur in year 0 and year 1. The early exercise checks are made, starting in year 1, each time a new present value is computed. Consider the year 1 node in which the asset price is 61.07. The present value of the expected future value is 2.26 and represents the value of the option if left alive. The early exercise proceeds at this node are max(0,55 – 61.07) = max(0,–6.07) = 0. Clearly the put is worth more alive than dead. Now, consider the year 1 node where the asset price is 40.94. The put if left alive is 12.20, while the early exercise proceeds are max(0,55 – 40.94) = 14.06. Obviously, we are better off exercising. At such nodes, we replace the value of the option left alive with the early exercise proceeds. Applying this procedure each time a present value in computed in year 1, and then again in year 0, we find that the current value of the American-style put option is 7.48. The American-style put option values at each node are shown in Figure 9.8. Note that the European-style put option value in Figure 9.7 is 6.64, and the American-style option value in Figure 9.8 is 7.48. The value of the privilege of being able to exercise this option early (i.e., the *early exercise premium*) appears to be about 84 cents.

We now use the JR parameters to determine the value of the put and isolate any differences. The up-step and down-step coefficients are given by (9.12a) and (9.12b). Substituting the problem information and raising the values to the exponent of e, we get the up-step and down-step coefficients of the asset price, that is,

$$u = e^{v} = e^{(b - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} = e^{(0.03 - 0.5(0.20^2))1 + 0.20\sqrt{1}} = 1.2337$$

and

$$d = e^{w} = e^{(b - 0.5\sigma^2)\Delta t - \sigma\sqrt{\Delta t}} = e^{(0.03 - 0.5(0.20^2))1 - 0.20\sqrt{1}} = 0.8270$$

FIGURE 9.8 Valuing an American-style put option using the Cox-Ross-Rubinstein parameters.

Time	0	1	2
			0.00
			74.59
		2.26	
		61.07	
	7.48		**5.00**
	50.00		50.00
		14.06	
		40.94	
			21.48
			33.52

FIGURE 9.9 Valuing an American-style put option using the Jarrow-Rudd parameters.

Year	0	1	2
			0.00
			76.10
		1.90	
		61.68	
	7.40		**3.99**
	50.00		51.01
		13.65	
		41.35	
			20.81
			34.19

Applying these coefficients to the current price of 50 generates the asset price lattice shown in Figure 9.9. Note that when the asset has an up-step followed by a down-step, it does not return to its original level. This is because JR set the up-step and down-step probabilities equal to 1/2. Consequently, any expected drift in the asset price through time must be handled through a lattice that drifts upward or downward. In contrast, the CRR procedure had an up-step followed by a down-step that returned the asset to its original price. Thus, the expected drift in asset price must be handled through the up-step and down-step probabilities. Recall that (9.8) shows that the probability of an up-step is greater than 1/2 as long as the asset price is expected to drift upward.

Figure 9.9 also contains the American-style put option values. They are computed using the same three-step procedure that we applied earlier. The only difference, as already noted, is in the definition of the values of u, d, and p. Note that the current value of the put is 7.40, where its value under the CRR parameters is 7.48. The difference is attributable to approximation error, and will tend to disappear as the number of time steps is increased.

The steps in the binomial method are summarized in Table 9.1. The main intuition underlying the procedure is that we can construct a discrete binomial asset price distribution whose jump sizes (u and d) and probabilities (p and $1 - p$) generate a mean and a variance equal to the mean and variance of the BSM continuous log-normal asset price distribution over the time increment, Δt. The combinations of jump sizes and probabilities are not unique. Table 9.1 also contains three different, commonly used parameter possibilities.[4]

Valuing a Barrier Option

The binomial method not only is straightforward to apply but also is very flexible in terms of the numbers and types of options that it can value. In Chapter 6, we introduced barrier options. Barrier options are of two types—“knock-out” and “knock-in.” A knockout option is like a standard option except that the

[4] Yet another possibility is given in Rendleman and Bartter (1979).

TABLE 9.1 Three steps in applying the binomial approximation method.

1. *Create asset price lattice.* Divide the option's time to expiration T into n increments of length Δt, that is, $\Delta t = T/n$. Start at the current level of asset price, S, and generate an asset price lattice by allowing the asset price jump up (down) by proportion $u(d)$. At the end of the first time increment, there will be two asset price nodes, Su and Sd. At the end of two time increments, there will be three asset price nodes, and so on. The final column in the lattice will have $n + 1$ nodes. The lattice is meant to capture all possible paths that the asset price may travel through the life of the option. The number of paths increases with n.
2. *Value option at expiration.* Value the option at expiration for each asset price node. The option value is 0 or the exercise proceeds, whichever is greater.
3. *Value option at earlier nodes by taking the present value of the expected future value.* Work your way back through time, one increment Δt at a time, by taking the present value of the expected future value of the option based on the two option nodes directly to the right of the valuation node. The expected values are computed using p, the probability of an up-step, and $1 - p$, the probability of a down-step. With each present value computation, check for any "boundary" violations (e.g., the early exercise boundary of an American-style option). When the recursive procedure is arrives back at time 0, the current value of the option is found.

The application of the binomial method requires values for the parameters u, d, and p. These are found by equating the mean and the variance of the discrete binomial distribution to the mean and the variance of the continuous log-normal distribution. Many combinations of parameters are possible. Three possibilities follow:

Method	u	d	p
Cox-Ross-Rubinstein (1979)	$e^{\sigma\sqrt{\Delta t}}$	$e^{-\sigma\sqrt{\Delta t}}$	$\frac{1}{2} + \frac{1}{2}\left(\frac{b - 0.5\sigma^2}{\sigma}\right)\sqrt{\Delta t}$
Jarrow-Rudd (1983)	$e^{(b-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}}$	$e^{(b-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}}$	1/2
Simple	$e^{\sigma\sqrt{\Delta t}}$	$e^{-\sigma\sqrt{\Delta t}}$	$\frac{e^{b\Delta t} - d}{u - d}$

option terminates if the asset price goes below or above some prespecified knock-out level. A knock-in option, on the other hand, is like a standard option except that it becomes "alive" only if the asset price goes below or above the knock-in barrier.

To illustrate the valuation of a barrier option, we modify the terms of the American-style FX put. Instead of assuming that the option is a *standard* American-style put, we assume that it is an *up-and-out* American-style put with a knock-out barrier of 60. In other words, this put terminates (i.e., expires worthless) if the asset price rises above 60 at any time during the option's life.

The steps of the binomial valuation are exactly as outlined above, except for Step 3(a). In addition to checking each node in year 1 for early exercise, we check if the knock-out condition applies. The upper node in year 1 has an asset price of 61.07, so the put is "knocked out" and its value is set equal to 0, as

FIGURE 9.10 Valuing an American-style "knock-out" FX put option using the binomial method.

Time	0	1	2
			0.00
			74.59
		0.00	
		61.07	
	6.35		**5.00**
	50.00		50.00
		14.06	
		40.94	
			21.48
			33.52

shown in Figure 9.10. The lower node in year 1 has an asset price of 40.94, so the knock-out condition does not apply. Taking the present value of the expected future value standing in year 0 then tells us that the value of this American-style, knock-out put is 6.34. Note that adding the knock-out feature reduces the value of the put by 1.13. This may help explain their popularity. If you are completely entirely convinced that the currency price will not rise, why pay for the extra insurance?

Assessing the Degree of Accuracy

The decision to use two time steps in the above illustrations was made only for expositional convenience. Fact of the matter is that such a crude grid provides a poor approximation of the option value. The procedure as outlined above, however, is perfectly general. We can set the number of time steps equal to any number we like. As the number of time steps is increased, the asset price lattice becomes more dense, with exponentially more price paths being considered. The number of price paths in the binomial model is 2^n. At two time steps, it was easy to see that the number of asset price paths over the life of the option is four. At 20 time steps, the number of paths is well over a million.

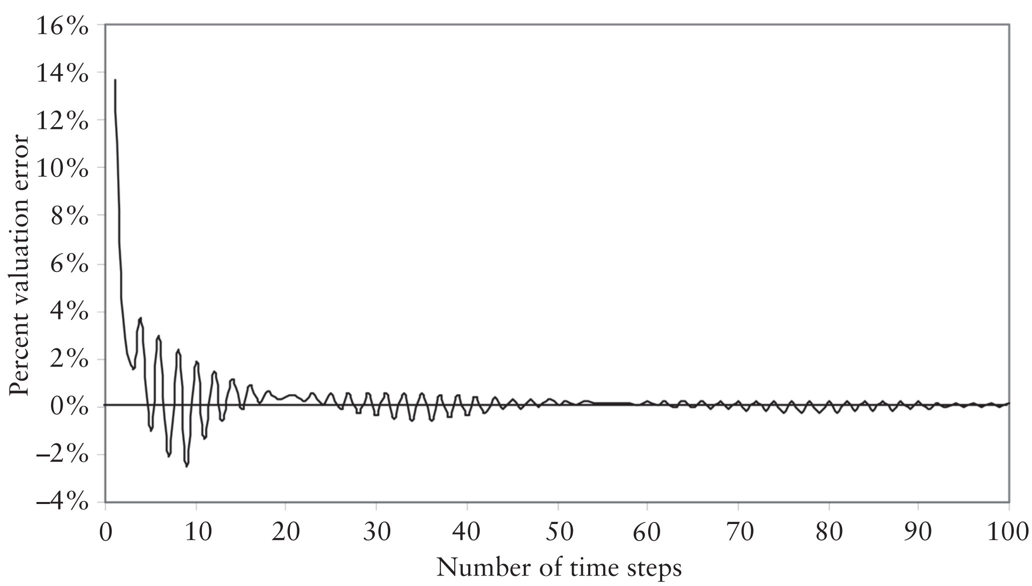
The decision regarding the appropriate number of time steps to use in the binomial method is therefore a cost/benefit analysis. The cost is computational time; the benefit is increased precision in valuation and risk measurement.[5] To gauge the degree of approximation error in the binomial method for a particular application, we can compare its results to the results of an option valuation problem whose valuation equation is known. Assume, for example, that the FX put

[5] The need for frequent time steps is particularly acute in the valuation of American-style barrier options. If the barrier price does not happen to coincide with nodes in the asset lattice, the monitoring of the barrier and, hence, option valuation will be inaccurate. Boyle and Lau (1994) show how to adjust the number of time steps in the binomial method so that the barrier price falls exactly on or very close to node values.

option in our illustration is European-style. Its value using the BSM option valuation formula (5.32) is 6.41. This is the true value and serves as our benchmark. Now, we apply the binomial method again and again increasing the number of time steps from 1 to, say, 100. At one time step, the value is 7.28, which represents a valuation error of 87 cents—13.67% of the true option value. At two time steps, the value is 6.64, as we established earlier in the chapter. This represents a valuation error of 23 cents[6]—3.70% of the true option value. As we increase the number of time steps, valuation precision increases albeit not monotonically. Figure 9.11 shows the pattern. The valuation error is –6 cents (–1.10%) at five time steps, 12 cents (1.89%) at 10 time steps, and so on. By 50 time steps, the absolute relative valuation error generally stays below 0.25%.[7]

The cost of increasing the number of time steps is computational time. From the description of the computational procedure, which is summarized in Table 9.1, it is fairly obvious that computational cost increases in direct proportion to the number of nodes in the lattice (i.e., the same set of computations is performed at each node prior to the option's expiration date). The number of nodes

FIGURE 9.11 Percent valuation error of the CRR binomial method as a function of the number of time steps.



[6] The reader can verify these figures using the OPTVAL Excel Function Library—OV_OPTION_VALUE provides the BSM model values and OV_APPROX_STD_OPT_BIN, provides the CRR and JR binomial method values.

[7] The fact that the valuation error oscillates from even to odd numbers of times steps is useful in designing more computationally efficient valuation procedures. With 30 time steps, for example, the number of nodes is 496 and the valuation error is –0.38%, and, with 31 time steps, the number of nodes is 528 and the valuation error is 0.58%. Thus, in this illustration, it is computationally cheaper and more accurate to average the values of the option obtained using 30 and 31 time steps (i.e., valuation error of 0.10%) than to use 50 times steps, where the number of nodes is 1,326 and the valuation error is about 0.25%.

in the lattice, on the other hand, increases at an increasing rate with the number of time steps, that is,

$$\text{Number of nodes} = \frac{(n+1)(n+2)}{2}$$

With two time steps, the number of nodes is six, with three time steps 10, with four time steps 15, and so on, as is illustrated in Figure 9.11. The question is where to set n. At 20 time steps, for example, the number of nodes is 231 and the relative pricing error falls in the range of ±0.5%. At 50 time steps, the number of nodes is 1,326 and the relative pricing error falls in the range of ±0.25%. Thus, to achieve increased relative valuation precision of 0.25%, we incurred 5.75 times the computational cost. Was it worth it?

There is no one answer to the question. It depends on the nature of the available computational resources and the importance of accuracy. The cost issue has become less important through time thanks to *Moore's law.* In April 1965, Gordon Moore, an engineer and cofounder of Intel, predicted that integrated circuit complexity would double every two years. His prediction has been surprisingly accurate. Today, the processing speed of a typical PC is more than 1,000 times faster than 20 years ago, while the cost of the PC is about a third. The accuracy issue is largely one of contract size. In terms of our European-style put illustration, a 0.25% valuation error amounts to less than 2 pennies (i.e., $6.41 \times 0.0025 = 0.0160$), hardly an amount worthy of concern. But, if the number of units of the underlying currency in the contract is 100 million rather than 1 (which has been implicitly assumed all along), however, the error is $1.6 million, an amount large enough to buy a supercomputer.

ILLUSTRATION 9.1 Assess degree of accuracy of competing valuation methods.

Consider a two-year European-style put option with an exercise price of 55. The underlying asset is assumed to be a foreign currency whose current price is 50 and whose volatility rate is 20%. The domestic rate of interest is assumed to be 5%, and the foreign rate of interest is 2%. Compare the performance of the binomial method using the CRR parameters with the binomial method using the JR parameters. Which is more accurate, holding the number of time steps constant, and why?

To make this assessment, we will first value the put analytically using the BSM formula. From the OPTVAL Library, we know

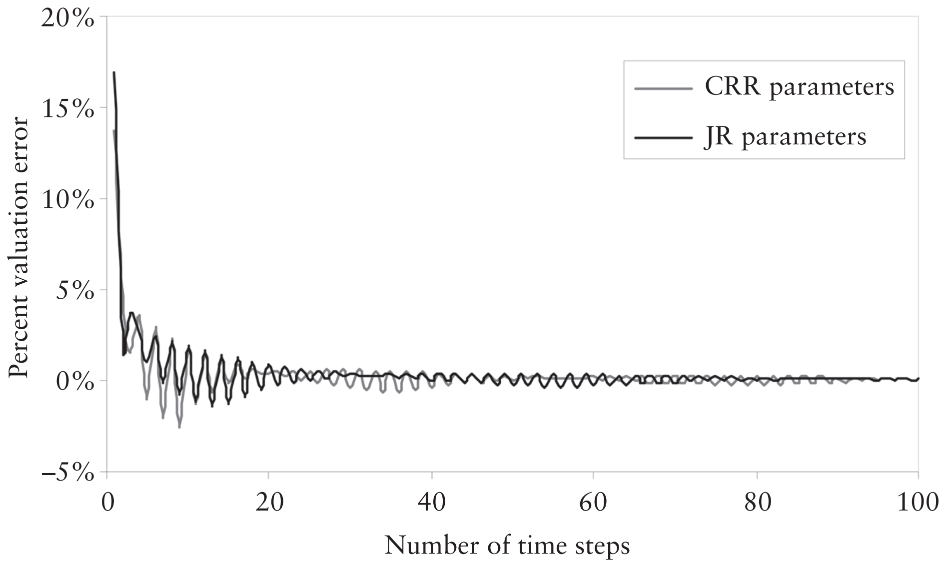
OV_OPTION_VALUE(50, 55, 2, 0.05, 0.02, 0.20, "p", "e") = 6.41

Because the put is European-style, we know that the analytical value is correct.

To assess the degree of accuracy of the competing binomial methods, we will use the OPTVAL function,

OV _APPROX_STD_OPT_BIN(50, 55, 2, 0.05, 0.02, 0.20, *n*, "p","e", *mthd*)

In using the function, we will vary the number of time steps n from 1 to 100. The binomial method algorithm uses the CRR parameters where *mthd* is set equal to 1 and the JR parameters where *mthd* is set equal to 2. The percent valuation error (relative to the analytical value) is then computed and plotted as a function of the number of time steps. The results are as shown below. The JR parameters appear to perform better than the CRR parameters. The oscillations in the percent valuation error are smaller, particularly for small numbers of time steps, and disappear almost completely for high numbers of time steps. The reason that the JR parameters produce a more accurate value is that they are based on the correct variance constraint in linking the means and the variances of the binomial distribution with the normal distribution of the logarithm of asset prices, that is, equation (9.4a). The CRR method uses only an approximation (9.4b). The valuation difference disappears as the number of time steps is increased (i.e., $\Delta t \to 0$).



Incorporating Discrete Flows

Thus far, we have addressed American-style option valuation when all of the carry costs of the asset underlying the option can be modeled as continuous rates. Recall that this assumption is most appropriate for foreign currencies and widely diversified stock portfolios. For common stock options or bond options, however, the dividend and coupon payments are best modeled as discrete cash flows. In such cases, the binomial method must be modified to account for these costs. The changes to the methodology are relatively minor, however. The intuition is that, if the amount and the timing of the discrete cash flows are known, the uncertainty regarding asset price is the uncertainty of the asset price net of the present value of the known cash flows. The steps in the binomial method are modified as follows.

In Step 1, create a lattice in terms of the asset price net of the present value of the income payments, that is, $S_0^x = S_0 - PVI_0$. This entails replacing the current asset price S_0 with S_0^x, since all subsequent values of the asset price are determined by applying the factors u and d to the current price. Note that the up-step and down-step coefficient u and d are computed using $b = r$, since the only *continuous* carry cost is the interest rate.

Step 2 remains unaltered. At time T, all of the income payments on the underlying asset made during the option's life have been made, and the lattice prices represent actual asset prices.

In Step 3, we compute the present value of the expected future values as before. The only distinction here is that we must adjust the early exercise bound to reflect the present value of any dividends paid between the valuation date at which we stand and the expiration date. For an American-style call, the early exercise proceeds are $\max(0, S^x_{i,j} + PVD_i - X)$, where $S^x_{i,j}$ is the lattice price at node (*i,j*), that is, at time *i* and asset price *j*, and PVD_i is the present value of all income payments between time *i* and expiration at time *T*. Note that at time 0, the early exercise proceeds of the call for the single remaining node are $\max(0, S^x_0 + PVD_0 - X)$, where $S^x_0 + PVD_0$ equals the current asset price *S* by the way we constructed the lattice.

ILLUSTRATION 9.2 Value American-style call option on dividend-paying stock.

Suppose that you own an American-style call option on a dividend-paying-stock. The call has 14 days remaining to expiration and an exercise price of \$55. The current stock price is \$60, and the volatility rate is 40%. The stock promises to pay a \$1 cash dividend in seven days. The risk-free rate of interest is 5%. Compute the value of the call using the binomial method with CRR parameters. Use two time steps.

First, identify the parameters of the binomial method implementation. The up-step and down-step coefficients are

$$u = e^{0.40\sqrt{7/365}} = 1.0570 \quad \text{and} \quad d = \frac{1}{u} = \frac{1}{1.0570} = 0.9461$$

the up-step probability is

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{0.05 - 0.5(0.40)^2}{0.40}\right)\sqrt{7/365} = 0.4948$$

and the down-step probability is 0.5052.

Second, compute the stock price net of the present value of the promised dividend, and create the stock price grid. The stock price net of the present value of the dividend is $60 - 1e^{-0.05(7/365)} = 59.001$. The stock price after two up-steps, for example, is $59.001(1.0570)^2 = 65.913$.

Third, compute the European-style call price by recursively taking the present value of the expected future value. The value of the European-style call is 4.66, as demonstrated in the lattice below. To illustrate the recursive computations, the value of the call at the upper node on day 7 is computed as

$$PVEFV = e^{-r\Delta t}EFV = e^{-0.05(7/365)}[0.4948(10.913) + 0.5052(4.001)] = 7.414$$

Days	0	7	14
			10.913
			65.913
		7.414	
		62.361	
	4.663		**4.001**
	59.001		59.001
		1.978	
		55.822	
			0.000
			52.813

Finally, to compute the value of the American-style call option, we must consider the effects of possible early exercise at the upper and lower nodes on day 7. At the upper node, compare the computed value, 7.414, with the value if exercised, 62.361 + 1 – 55 = 8.361. Since the early exercise proceeds are higher, replace the computed option value at this node with the early exercise proceeds.[8] At the lower node the, computed value, 1.978, exceeds the early exercise proceeds, 55.822 + 1 – 55 = 1.822, so no replacement is made. The value of the American-style call is as shown below. The value of the American-style call is 5.132, hence the value of the early exercise premium is 0.469.

Days	0	7	14
			10.913
			65.913
		8.361	
		62.361	
	5.132		**4.001**
	59.001		59.001
		1.978	
		55.822	
			0.000
			52.813

Two Underlying Sources of Risk

The binomial method can also be extended to handle multiple sources of risk. As discussed in Chapter 6, options on the minimum and the maximum of two risky assets qualify. A call option on the maximum, for example, has a payoff $\max[0, \max(\tilde{S}_{1,T}, \tilde{S}_{2,T}) - X]$ at the option's expiration. Stulz (1982) shows that if options on the minimum and maximum of two risky assets are European-style, they can be valued analytically.[9] If they are American-style, however, they must be valued numerically. Similarly, both European- and American-style spread options must be valued numerically. A European-style call option on a spread, for example, has a payoff $\max[0, \tilde{S}_{1,T} - \tilde{S}_{2,T} - X]$ at the option's expiration. Since asset prices are log-normally distributed under the BSM assumptions, the price difference, $\tilde{S}_{1,T} - \tilde{S}_{2,T}$, is not log-normally distributed and, hence, the usual BSM valuation mechanics cannot be applied.

Boyle, Evnine, and Gibbs (1989) modify the CRR binomial method to handle multiple sources of risk. We consider only the two-asset case. Like in the CRR framework, each asset's price is allowed to jump up or down. Hence, at any given instant, four jumps are possible—up-up, up-down, down-up and down-down, where the first move in each pair of movements is for asset 1 and the second is for asset 2. The probabilities of each pair of movements are p_1 through p_4, respectively. The proportionate jumps in asset price are the same as those for the CRR formulation, that is,

[8] The motivation for early exercise is being driven by the fact, if the call option holder waits until expiration to exercise, he will have forfeit the opportunity to receive the dividend.

[9] Johnson (1987) shows how the Stulz (1982) results can be extended to the case of multiple underlying assets.

$$u_i = e^{\sigma_i\sqrt{\Delta t}} \text{ and } d_i = 1/u_i, \text{ for } i = 1, 2 \tag{9.13}$$

The probabilities of the different pairings are

$$p_1 = \frac{1}{4}\left[1 + \rho + \sqrt{\Delta t}\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right)\right] \tag{9.14a}$$

$$p_2 = \frac{1}{4}\left[1 - \rho + \sqrt{\Delta t}\left(\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\right)\right] \tag{9.14b}$$

$$p_3 = \frac{1}{4}\left[1 - \rho + \sqrt{\Delta t}\left(-\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right)\right] \tag{9.14c}$$

and

$$p_4 = \frac{1}{4}\left[1 + \rho + \sqrt{\Delta t}\left(-\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\right)\right] \tag{9.14d}$$

Under the assumption of risk-neutrality, the mean continuously compounded rate of price appreciation equals the asset's risk-neutral cost of carry rate, that is, $\mu_i = b_i - 0.5\sigma_i^2$, for $i = 1, 2$.

ILLUSTRATION 9.3 Value spread option using binomial method.

Compute the value of a American-style call option on the "crack" spread between the prices of an unleaded gasoline futures and a crude oil futures[10] *using the binomial method. Assume the gasoline futures has a price of \$22 per barrel and a volatility rate of 30% annually, and the crude oil futures has a price of \$20 per barrel and a volatility rate of 20%. The correlation between the rates of price appreciation for the two futures contracts is 0.85. The option has an exercise price of \$2 and a time to expiration of three months. The risk-free rate of interest is 5%.*

The three steps of the binomial method are summarized in Table 9.1. Since the mechanics of setting up a two-dimensional price lattice are cumbersome, we will simply apply the appropriate valuation approximation from the OPTVAL function library. The syntax of the function call is

OV_APPROX_SPRD_OPT_BIN(*s1, s2, x, t, r, i1, i2, v1, v2, rho, n, cp, ae*),

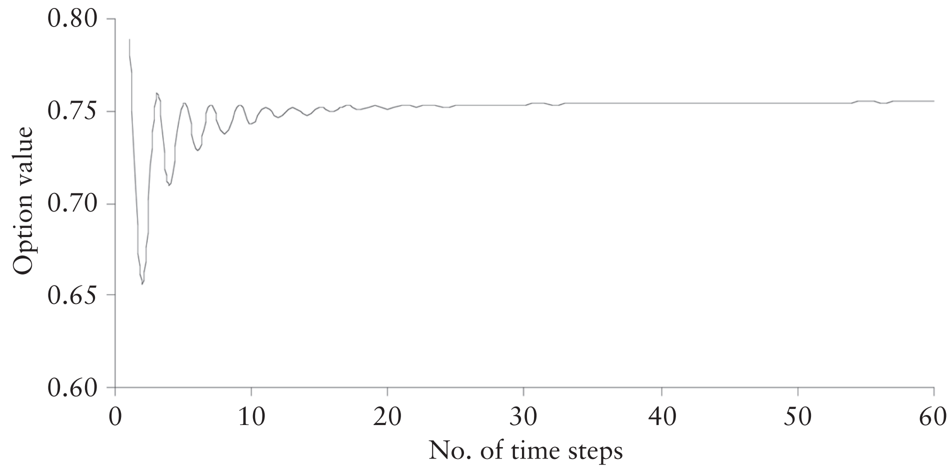
where *s1* and *s2* are the underlying asset prices (with the spread defined as *s1 – s2*), *x* is the exercise price of option, *t* is the option's time remaining to expiration, *r* is the risk-free interest rate, *i1* and *i2* are the income rates of assets 1 and 2, *v1* and *v2* are the volatility rates of

[10] Crack spread futures options trade on the New York Mercantile Exchange (NYMEX). The parameters for this illustration are drawn from Whaley (1996).

assets 1 and 2, *rho* is the correlation between asset returns, *n* is the number of time steps, *cp* is a call/put indicator ("C" or "c" for call and "P" or "p" for put), and *ae* is an American- or European-style option indicator ("A" or "a" for American-style and "E" or "e" for European-style). Using 25 time steps, the value of the call is

OV_APPROX_SPRD_OPT_BIN(22, 20, 2, 0.25, 0.05, 0.05, 0.05, 0.30, 0.20, 0.85, 25, "c","e") = 0.7519.

Note that the risk-free interest rate is used as the income rate for both assets. We tricked the valuation algorithm into thinking the underlying assets are futures contracts by implicitly setting the net cost of carry rate to zero.[11] Like in the univariate case, the accuracy of the bivariate binomial method improves with the number of times steps. For the valuation parameters in this illustration, little variation in value remains after 20 time steps.



TRINOMIAL METHOD

The trinomial method has the same three steps as the binomial method. The only difference is that in place of allowing the asset price to go only up or down from its current price, the asset price can go up, down, or stay the same. The definitions of the up-step and down-step coefficients change, as do the definitions of the probabilities.

Under the trinomial method, we set the mean and the variance of a discrete trinomial distribution for the logarithm of asset price equal to the corresponding mean and variance of the continuous normal distribution of the logarithm of asset price. The mean constraint is

$$p_u v + p_d(-v) = \mu\Delta t \tag{9.15}$$

where p_u is the probability of an up-step, and p_d is the probability of a down-step. Condition (9.15) is the trinomial method's counterpart to the mean constraint (9.2) in the development of the binomial method. Note that we are assuming that

[11] Recall that the Black (1976) futures option valuation formula in Chapter 7 was a special cash of the BSM formula (7.32) where the cost of carry rate b was set equal to zero.

the absolute size of the up-step and the down-step are the same. In this sense, we have imposed the CRR symmetry restriction. The probability of no change in the asset price is $p_m = 1 - p_u - p_d$ and does not enter (9.15) since the price change is zero. In our implementation of the trinomial method, the variance constraint is

$$v^2(p_u + p_d) = \sigma^2 \Delta t \tag{9.16}$$

This is the counterpart to the binomial method's variance constraint (9.4b) in which the higher order terms of Δt are ignored. While ignoring higher order terms simplifies matters, a further restriction on the parameters is necessary in order to make the model usable since we have three unknowns—p_u, p_d, and v, and only two equations—(9.15) and (9.16).

The final restriction is drawn from the work of Boyle (1988a) and Kamrad and Ritchken (1991) (hereafter, "KR"). They assume the up-step coefficient has the functional form, $v = \lambda\sigma\sqrt{\Delta t}$, where $\lambda \geq 1$. Substituting into (9.16), we find that the sum of the probabilities of an up-step and a down-step is

$$p_u + p_d = 1/\lambda^2 \tag{9.17}$$

The probability of no change in price is therefore

$$p_m = 1 - 1/\lambda^2 \tag{9.18a}$$

Isolating the probability of a down-step in (9.17) and then substituting into (9.15), shows that

$$\begin{aligned} p_u &= \frac{1}{2\lambda^2} + \frac{\mu\Delta t}{2\lambda\sigma\sqrt{\Delta t}} \\ &= \frac{1}{2\lambda^2} + \frac{1}{2\lambda}\left(\frac{b - 0.5\sigma^2}{\sigma}\right)\sqrt{\Delta t} \end{aligned} \tag{9.18b}$$

and therefore

$$\begin{aligned} p_d &= \frac{1}{2\lambda^2} - \frac{\mu\Delta t}{2\lambda\sigma\sqrt{\Delta t}} \\ &= \frac{1}{2\lambda^2} - \frac{1}{2\lambda}\left(\frac{b - 0.5\sigma^2}{\sigma}\right)\sqrt{\Delta t} \end{aligned} \tag{9.18c}$$

Where $\lambda = 1$, note that the trinomial model collapses to the CRR binomial model. The probability of a zero price change is 0, so the middle node drops out. The up-step coefficient in asset price is

$$u = e^{v} = e^{\lambda\sigma\sqrt{\Delta t}} = e^{\sigma\sqrt{\Delta t}}$$

exactly as in CRR, and the probability of an up-step within the trinomial framework (9.18b) equals the probability of an up-step in the binomial framework (9.6).[12]

The choice of an appropriate value of λ is left to the user. The higher the value of λ, the greater is the probability that the asset price will move sideways rather than up or down. In the application below, we set λ equal to the square root of 2. At $\lambda = \sqrt{2}$, the probability of the middle step is $p_m = 1 - 1/\lambda^2 = 1/2$.

Applying the Trinomial Method

Applying the trinomial method has the same three steps as the binomial method. To illustrate its use, we will value a two-year, American-style FX put option with an exercise price of 55. The current exchange rate is 50, and its volatility rate is 20%. The domestic rate of interest is assumed to be 5%, and the foreign rate of interest, 2%. The expected risk-neutral rate of price appreciation of the currency is 3%.

Step 1: Create the Asset Price Lattice Like in the case of the binomial method, the first step in the trinomial method is to set up the asset price lattice. If the current asset price is S, the asset price may jump only up to a level of uS (where $u > 1$), down to a level of dS (where $d < 1$), or horizontally to the level S. The CRR restriction $ud = 1$ has been assumed. Setting $\lambda = \sqrt{2}$, the value of u is

$$u = e^{\sigma\sqrt{2\Delta t}} = e^{0.20\sqrt{2}} = 1.3269$$

and the value of d is $d = 1/u = 0.7536$. Applying these coefficients to the current asset price generates the two-period lattice shown in Figure 9.12. Note that the tree is denser than the binomial tree. This stands to reason since the number of branches from each node is three instead of two. The range of terminal asset prices at the option's expiration is also greater.

FIGURE 9.12 Two-period trinomial asset price lattice for valuing an option.

Time	0	1	2
			88.03
		66.34	66.34
	50.00	50.00	50.00
		37.68	37.68
			28.40

[12] Rubinstein (2000) discusses the relation between the binomial and trinomial option pricing models.

FIGURE 9.13 Valuing an American-style put option using a two-period trinomial method.

Year	0	1	2
			0.00
			88.03
		1.10	**0.00**
		66.34	66.34
	7.06	**6.21**	**5.00**
	50.00	50.00	50.00
		17.32	**17.32**
		37.68	37.68
			26.60
			28.40

Step 2: Value the Option at Expiration The second step also parallels the binomial method, that is, we value the option at expiration. At expiration (i.e., where $i = T$), the value of the option at node j is $\max(0, X - S_{i,j})$. The numerical values of the put at expiration are shown in Figure 9.13.

Step 3: Value Option at Earlier Nodes by Taking the Present Value of the Expected Future Value The next step is again similar to the binomial method in that we take the present value of the expected future value in an iterative fashion. The probability of the middle step is 1/2, as was noted earlier. The probabilities of an up-step and a down-step are computed using (9.18b) and (9.18c), that is,

$$p_u = \frac{1}{2} + \frac{1}{2}\left(\frac{0.03 - 0.5(0.20^2)}{0.20}\right) = 0.2677$$

and

$$p_u = \frac{1}{2} - \frac{1}{2}\left(\frac{0.03 - 0.5(0.20^2)}{0.20}\right) = 0.2323$$

We now compute the present value of the expected future value at each node in the tree. Consider the asset price at the highest node in year 1, 66.34. The present value of the expected future value of the put is

$$PVEFV = e^{-0.05\sqrt{1}}[0.2677(0.00) + 0.5000(0.00) + 0.2323(5.00)] = 1.10$$

Before proceeding to the next node, we check the early exercise condition. Since the put is out of the money at this node, we leave it alive. At the year 1 asset price node of 50,

$$PVEFV = e^{-0.05\sqrt{1}}[0.2677(0.00) + 0.5000(5.00) + 0.2323(17.32)] = 6.21$$

At an asset price of 50, the put has exercise proceeds of 5. Since its value if left alive is 6.21, early exercise is not optimal and we do not replace the *PVEFV* with the exercise proceeds. At the year 1 asset price node of 37.68,

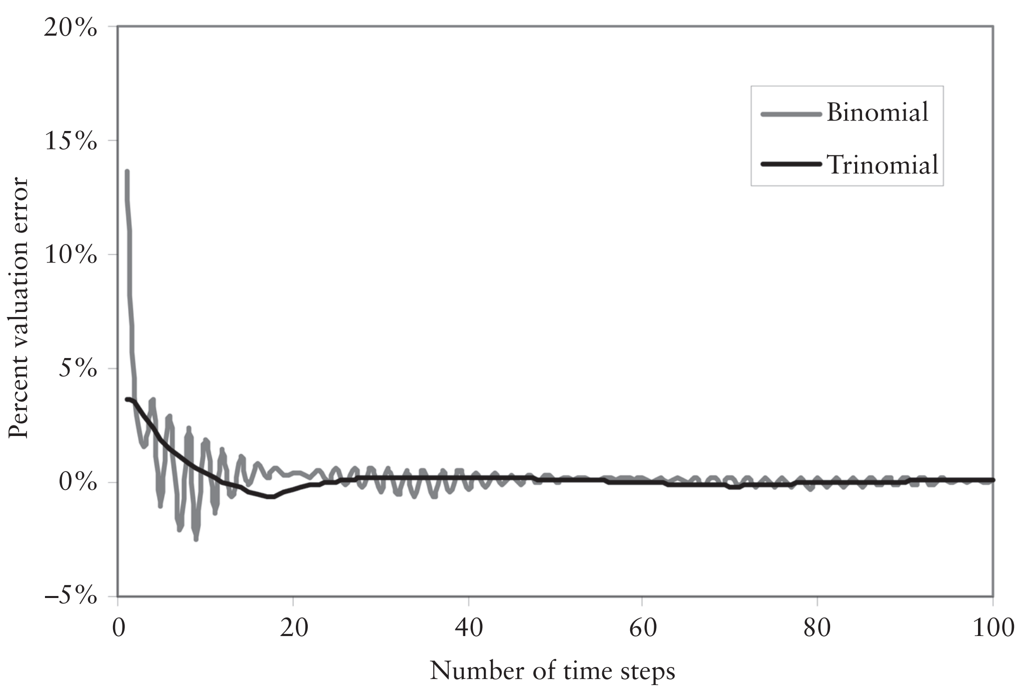
$$PVEFV = e^{-0.05\sqrt{1}}[0.2677(5.00) + 0.5000(17.32) + 0.2323(26.60)] = 15.39$$

Here, the put has exercise proceeds of 17.32, which exceed the value of the put if left alive. Thus, we replace the present value, 15.39, with the exercise proceeds, 17.32, and move to the next time step. The value of the American-style put option using the trinomial lattice with two time steps is 7.06. The value of the European-style put is 6.63.

Assessing the Degree of Accuracy

Like the binomial method, the accuracy of the trinomial method improves with the number of time steps. Indeed, it does so at a much quicker rate, as Figure 9.14 shows. At ten time steps, the trinomial method appears to produce option values as accurate as the binomial method at 20. This should not be surprising. Recall that the binomial method produces $(n + 1)(n + 2)/2$ nodes and considers

FIGURE 9.14 Approximation error of the binomial and trinomial approximation methods as a function of the number of time steps.



2^n asset price paths over the life of the option. For the same number of time steps, the trinomial method produces $(n + 1)^2$ nodes and considers 3^n price paths. At 10 time steps, the binomial method has 66 nodes and incorporates 1,024 possible asset price paths. The trinomial method, on the other hand, has 121 nodes and incorporates 59,049 price paths. But the increased accuracy of the trinomial method comes at a cost. Recall that computational cost varies directly with the number of nodes computed. For the 10-time step example, the trinomial method costs roughly double the binomial method.

MONTE CARLO SIMULATION

Monte Carlo simulation techniques are also used to value derivative contracts.[13] Like the lattice-based procedures, the technique involves simulating possible paths that the asset price may take over the life of the option. Unlike the lattice-based procedures that trace out all possible asset price paths at the outset, the Monte Carlo technique produces a price path adding up a series of randomly drawn price increments over the life of the option. Each drawing corresponds to the time increment Δt, and a series of n drawings produces a *simulation run* (i.e., an asset price path). The Monte Carlo technique involves repeated simulation runs or trials. To value a European-style put option, for example, each trial produces a terminal asset price, which, in turn, is used to determine the terminal option value. After, say, 10,000 trials, the terminal options values are averaged arithmetically to obtain the expected terminal option value, $E(\tilde{p}_T)$, and then the expected value is discounted to the present at the risk-free interest rate, $p = e^{-rT}E(\tilde{p}_T)$.

Geometric Brownian Motion

What remains is the description of how each asset price path is generated. Under the BSM model, the asset price follows a continuous Brownian diffusion process. Like in the case of the lattice-based procedures, we must replace this continuous process with movements over discrete intervals. By setting the number of time intervals during the life of the option, n, the time increment becomes $\Delta t = T/n$. To generate a movement over the interval, we draw a random number, ε, from a unit normal distribution.[14] This number is used to update the asset price at the beginning on the time interval. Recall that under the BSM assumptions, the change in the logarithm of asset price is normally distributed with mean, $(b - \sigma^2/2)\Delta t$, and

[13] Boyle (1988b) was the first to apply Monte Carlo simulation techniques to option valuation.

[14] Methods for generating univariate normal deviates (i.e., random numbers drawn from a normal distribution with mean 0 and variance 1) are available in most computer programming languages. In Excel, the task can be accomplished using the command, =NORMSINV(RAND()). The logic is as follows. NORMSINV is the inverse of the function NORMSDIST. Recall that we used NORMSDIST in Chapter 7 to measure the probability that a random number drawn from a unit normal distribution has a value below d. Thus if we have a random number drawn from a uniform distribution over the range from zero to one, we can insert it into the NORMSINV function to generate a random drawing from a unit normal distribution. The function, RAND() generates a random number from a uniform distribution whose range is zero to one.

standard deviation, $\sigma\sqrt{\Delta t}$.[15] To update the logarithm of asset price, therefore, we (a) scale the random drawing ε (which has a standard deviation of 1) by the standard deviation of the logarithm of asset price change, $\sigma\sqrt{\Delta t}$, (b) add it to the expected movement, $(b - \sigma^2/2)\Delta t$, and (c) add the sum to the beginning of period logarithm of asset price, $\ln S_t$, that is,

$$\ln S_{t+\Delta t} = \ln S_t + (b - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon \tag{9.19}$$

Like in the case of the lattice-based procedures, individuals working with the Monte Carlo technique may prefer to see the sequence of asset prices in the simulation run rather than the sequence of the logarithm of asset prices. In this case, the updating is accomplished using an equation created by raising both sides of (9.19) to the power of e, that is,

$$S_{t+\Delta t} = S_t e^{(b - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon} \tag{9.20}$$

Equation (9.20) transforms the unit normally distributed random variable ε into a log-normally distributed asset price.

Table 9.2 uses the parameters of our two-year put option illustration to show some of the computations performed in the first simulation run. We arbitrarily set the number of time steps to be equal to the number of days to expiration, 730. The first drawing from the unit normal distribution produced a value of –0.8369. Substituting into (9.20), we find that the asset price at the end of the first day is

TABLE 9.2 First simulation run in valuing a two-year European-style put option written on a currency price using Monte Carlo simulation.

Number of Time Step	Random Drawing from a Unit Normal Distribution, ε	Asset Price, $S_{t+\Delta t}$
0		50.0000
1	–0.8369	49.5652
2	–0.1723	49.4773
3	0.1871	49.5757
...	...	...
728	–0.9772	66.5649
729	0.1087	66.6425
730	–0.1798	66.5190

[15] Under the BSM assumptions, the change in the log of asset price, $\ln S_{t+\Delta t} - \ln S_t$, is normally distributed with mean $(b - \sigma^2/2)\Delta t$ and standard deviation, $\sigma\sqrt{\Delta t}$. Thus it follows that

$$\varepsilon = \frac{(\ln S_{t+\Delta t} - \ln S_t) - (b - \sigma^2/2)\Delta t}{\sigma\sqrt{\Delta t}}$$

has a unit normal distribution.

$$S_1 = 50e^{(0.03-0.20^2/2)(1/730)+0.20\sqrt{1/730}(-0.8369)} = 49.5652$$

The drawing on the second day, –0.1723, produces an asset price of

$$S_2 = 49.5652e^{(0.03-0.20^2/2)(1/730)+0.20\sqrt{1/730}(-0.1723)} = 49.4773$$

as so on. After the 730th drawing in the simulation run, the asset price is 66.5190. Since the put option has an exercise price of 55, the put finishes out-of-the-money and has a terminal value of 0. This completes the first trial.

The simulation run procedure is repeated 9,999 more times. Each time, the terminal value of the put is recorded. Table 9.3 summarizes the results across the 10,000 runs. The average terminal asset price is 53.1144. This closely corresponds, but is not exactly equal, to the expected terminal asset price computed based on the cost of carry rate, that is, $50e^{0.03(1)} = 53.0918$. The difference is attributable to sampling error in the simulation. Across the 10,000 trials, the terminal asset price ranged from 18.2019 to 139.7387, and the standard deviation of the terminal asset price was 15.4352. To measure the degree of potential error, we can compute the standard error of the estimate as

TABLE 9.3 Terminal asset price and terminal put option value in each of the 10,000 simulation runs performed for the European-style put option written on a currency price using Monte Carlo simulation.

Simulation Run	Terminal Asset Price	Terminal Put Value
1	66.5190	0.0000
2	40.8806	14.1194
3	59.5425	0.0000
…	…	…
9,998	50.4133	4.5867
9,999	30.7052	24.2948
10,000	51.8364	3.1636
Summary statistics		
Average	53.1144	7.1274
Std. deviation	15.4352	8.1269
Std. error	0.1544	1.1288
Minimum	18.2019	0.0000
Maximum	139.7387	36.7981
95% confidence interval		
Lower bound	52.8119	4.9150
Upper bound	53.4169	9.3398
Current values		
Present value of expected value	50.0213	6.4492

$$\begin{aligned}\text{Standard error of estimate} &= \frac{\text{Standard deviation of terminal values}}{\sqrt{\text{Number of trials}}}\\ &= \frac{15.4352}{\sqrt{10{,}000}} \qquad (9.21)\\ &= 0.1544\end{aligned}$$

Thus, the 95% confidence interval for the estimate is

$$53.1144 - 1.96(0.1544) = 52.8119 \le S_T \le 53.4156 = 53.1144 + 1.96(0.1544)$$

Although the Monte Carlo procedure is imprecise, we are 95% confident that the terminal asset price will lie between 52.8119 and 53.4169.

The average terminal value of our European-style put is 7.1274, and its 95% confidence interval is 4.9157 to 9.3406. This range of terminal put option values is quite large. While increasing the number of trials increases the precision of the estimate of option value, the increased precision increases only with the square root of the number of trials, as is shown in the formula for the standard error of the estimate (9.21). Note that the current value of the option using 10,000 Monte Carlo simulation runs, 6.4492, is reasonably close to the value of the BSM European-style put option valuation equation, that is, 6.41.

A key advantage of the Monte Carlo method is that we can measure the degree of valuation error directly using the standard error of the estimate. Another advantage of the Monte Carlo technique is its flexibility. Since the path of the asset price beginning at time 0 and continuing through the life of the option is observed, the technique is well suited for handling a variety of nonstandard options whose payoff contingencies are well defined through time (e.g., European barrier options, and accrual options) and for simulating the performance of possible trading strategies. Yet another advantage is that it can be adapted easily to handle multiple sources of price uncertainty. The Monte Carlo technique's chief disadvantage is that it can be applied only when the option payout does not depend on the option's value at future points in time. This eliminates the possibility of valuing American-style options since the decision to exercise early depends on the value of the option that will be forfeit.

ILLUSTRATION 9.4 Using Monte Carlo simulation to capture effects of dynamic hedging.

Suppose that you own 2 million shares of ABC's stock and have just entered a costless collar agreement on your shares with an OTC options dealer. In the collar agreement, you are long a European-style put with an exercise price of \$30 a share, and you are short a European-style call with an exercise price of \$60 a share. Both of the options have one-year to expiration, and the agreement was consummated with no upfront payment. Compute the amount of the fee embedded in this OTC agreement, and use Monte Carlo simulation to demonstrate how the OTC dealer earns the fee. Set the time step equal to one month. Assume the risk-free rate of interest is 5%. Also assume that ABC's stock has an expected rate of return of 15%, a share price of \$45, a volatility rate is 40%, and no expected cash dividends.

The embedded cost of the collar may be computed using the European-style option valuation formulas from Chapter 7. Using the problem parameters, the values of the call and put are \$3.415 and \$0.899, respectively. Since you are short the call and long the put, you have implicitly paid a fee of \$2.516 per share on 2 million shares or \$5,032,042 in total. Assuming this amount is invested at the risk-free rate of interest, its value at the end of one year is \$5,290,040.

Collar Valuation	
Stock pricce	45.00
Put exercise price	30.00
Call exercise price	60.00
Time to expiration in years	1.00
Interest rate	5.00%
Expected stock return	15.00%
Volatility rate	40.00%
Put value	0.899
Call value	3.415
Call less put value	2.516
Number of shares	2,000,000
PV of embedded fee	5,032,042
FV of embedded fee	5,290,040

Under a costless collar agreement, no money changes hands at the outset. But, as your computations show, you have paid an implicit fee of \$5,032,042. Your counterparty, the OTC options dealer, has received this fee in the form of a long call/short put position valued at \$5,032,042. To lock-in this gain (i.e., to "monetize" the value of this trade), the dealer must dynamically hedge his option position. Since holding a long call and a short put is equivalent to holding the underlying stock, the dealer can hedge by shorting stocks so that any change in the value of the option position is offset by the change in the short stock position. The number of shares to short is determined by the collar's delta. The delta of the put is –0.0903 and the delta of the call is 0.3467. Thus, the dealer needs to sell 0.4371 shares of stock for each share in the agreement. The total number of shares that he will sell to hedge the collar agreement is 874,110. This generates \$39,334,962 in cash, which he promptly puts in risk-free bonds. The value of these bonds at the end of the year is \$41,351,708.

As you are aware, the initial hedge is risk-free only for the next instant in time and for infinitesimal movements in the stock price. Rebalancing continuously, however, is not practical since trading costs would be infinite. Here, we assume rebalancing takes place on a monthly basis. At the end of the first month, the dealer will rebalance his position to again make it delta-neutral. To simulate the value/risk of his position, he draws a unit normally distributed random variable and computes an end-of-month stock price.

$$S_1 = 45e^{(0.15 - 0.20^2/2)(1/12) + 0.40\sqrt{1/12}(-0.836854)} = 41.0941$$

He then computes the new deltas for the call and the put. Because the net delta of the position has fallen, he now needs fewer shares to hedge. He buys back 118,481 shares in the market at the prevailing price of \$41.0941, which costs \$4,868,864. He then carries that cost for 11 months, producing a terminal cash outflow of \$5,097,213.

At the end of the second month, he will rebalance his portfolio yet once again. To simulate the value/risk of his position, he draws another random variable, –0.172280, and updates the stock price,

$$S_2 = 41.0941e^{(0.15-0.20^2/2)(1/12)+0.40\sqrt{1/12}(-0.172280)} = 40.5204$$

He again computes the new deltas, and he finds that he needs yet fewer shares in his delta-hedge. He buys back another 56,512 shares at the prevailing market price of $40.5204. The cost is $2,289,905 at the end of month two, or $2,387,334 at the end of the options' lives.

The simulation is repeated again and again through month 12, and a summary is provided below. Note that at expiration, the amount of money in the cash account is $125,617,428. But, the market maker has 2 million shares shorted, which he has to cover. He buys the shares at the prevailing market price, $70.0197, which costs $140,039,368 in total. He then exercises his calls. Each call is in-the-money by $10.0197, so he earns $20,039,368 in total. The net terminal value across all of these values is $5,617,428. Based on the prices when the position was entered, the terminal value was expected to be $5,290,040. The difference, $327,388, is attributable to the fact that we performed only a single simulation run. The Excel file, **Delta hedge.xls**, contains the worksheet used to generate the table below. You will not get exactly the same values because your set of random drawings will be different. Nonetheless, it will help to reinforce the mechanics of the computations. Note that this spreadsheet may perform very slowly given the number of trials being executed:

Period	Random Draw	Closing Price	Years to Expiration	Put Delta	Call Delta	Net Delta	Aggregate Delta	Change in Delta	Shares Sold	Cash Paid/ Received	Terminal Value of Cash
0		45.0000	1.0000	–0.0903	0.3467	0.4371	874,110		–874,110	39,334,962	41,351,708
1	–0.836854	41.0941	0.9167	–0.1286	0.2492	0.3778	755,629	–118,481	118,481	–4,868,864	–5,097,213
2	–0.172280	40.5204	0.8333	–0.1314	0.2182	0.3496	699,117	–56,512	56,512	–2,289,905	–2,387,334
3	0.187117	41.6476	0.7500	–0.1096	0.2199	0.3295	659,100	–40,017	40,017	–1,666,620	–1,730,304
4	1.615440	50.4818	0.6667	–0.0315	0.3961	0.4276	855,205	196,105	–196,105	9,899,729	10,235,281
5	–0.176774	49.7512	0.5833	–0.0285	0.3576	0.3861	772,104	–83,100	83,100	–4,134,343	–4,256,703
6	0.653145	53.9623	0.5000	–0.0106	0.4423	0.4529	905,720	133,616	–133,616	7,210,216	7,392,744
7	–0.546364	50.9595	0.4167	–0.0119	0.3362	0.3481	696,203	–209,517	209,517	–10,676,871	–10,901,639
8	0.194146	52.4197	0.3333	–0.0046	0.3456	0.3502	700,428	4,225	–4,225	221,481	225,204
9	0.925709	58.6746	0.2500	–0.0002	0.5203	0.5205	1,040,961	340,533	–340,533	19,980,621	20,231,947
10	1.204321	67.8231	0.1667	0.0000	0.8114	0.8114	1,622,870	581,908	–581,908	39,466,841	39,797,106
11	1.530555	81.4077	0.0833	0.0000	0.9969	0.9969	1,993,786	370,917	–370,917	30,195,459	30,321,536
12	–1.355560	70.0197	0.0000	0.000	1.000	1.0000	2,000,000	6,214	–6,214	435,096	435,096
Totals									–2,000,000		125,617,428

	Shares Outstanding	Share Price Change	Terminal Value
Terminal value of cash account			125,617,428
Cover remaining shares outstanding	–2,000,000	70.020	–140,039,368
Bank exercises call option			20,039,368
Customer exercises put option			0
Net proceeds from selling collar			5,617,428

To determine the effectiveness of delta-hedging on a monthly basis on average, the simulation experiment as outlined above would be run again and again. In the table below, the simulation results are reported. With 1,000 simulation runs, the average terminal value is $5,191,907, much close to the expected terminal value of $5,290,040. The range of terminal values is incredibly large, however, from –$10,008,845 to $26,521,463. In all likelihood, the OTC dealer would find this level of risk unacceptable, and would rebalance more frequently:

	Mean	Standard Deviation	Minimum	Maximum	Average Number of Shares Traded	After Trading Cost Profit	Profit per Unit of Risk
Monthly	5,191,907	4,108,591	–10,008,845	26,521,463	3,135,707	4,878,336	1.187
Weekly	5,343,014	2,002,836	–631,657	13,943,237	5,192,995	4,823,714	2.408
Daily	5,285,976	786,634	2,174,783	8,786,527	12,085,420	4,077,434	5.183
Hourly (6)	5,290,299	301,906	4,316,409	6,852,635	27,957,722	2,494,527	8.263
Hourly (12)	5,296,894	214,420	4,418,705	6,416,575	38,567,903	1,440,104	6.716
Hourly (24)	5,304,550	151,603	4,646,940	6,116,745	54,896,398	–185,089	–1.221

With weekly rebalancing, the average terminal value across the 1,000 simulation runs is $5,343,014, even closer to the expected value of $5,290,040. Note also that the standard deviation of the terminal values across the 1,000 runs for weekly rebalancing is less than half the standard deviation for monthly rebalancing. With the monthly rebalancing, the average number of shares traded to hedge the collar over its life was 3,135,707. With weekly rebalancing, the average number was 5,192,995. Assuming trading costs of $.10 a share, the expected profit for weekly rebalancing remains higher. To gauge the performance on a risk-adjusted basis, expected after-trading cost value can be divided by the standard deviation of terminal value. The ratio for monthly rebalancing is 1.187 and the ratio for weekly rebalancing is 2.408, indicating the dominance of the weekly strategy.

The table also includes daily rebalancing as well as hourly rebalancing (assuming 6, 12 and 24 hours in the day). The more frequent the rebalancing, the lower the standard deviation of terminal values, however, the greater the trading costs. The maximum ratio of expected after-trading cost profit to risk is 8.263 and occurs for the simulation in which the hedge is rebalanced six times a day. Any risk reduction benefit from rebalancing more frequently is offset by incremental trading costs. Indeed, rebalancing each hour, 24 hours a day, produces an expected after-trading cost of –$185,089.

ILLUSTRATION 9.5 Value average rate option.

An average rate option (sometimes referred to as an Asian-style option) is an option whose payoff is based on an arithmetic average[16] of the underlying asset price during the option's life. In some instances, the exercise price is fixed, and the average asset price is used as the terminal asset price. In other instances, the average asset price is used as the exercise price and is compared to the terminal asset price to determine the option's payoff.[17] Monte Carlo simulation is a useful tool in valuing these path dependent options.

[16] Less frequently, Asian options are based on a geometric average. While using a geometric average makes the valuation problem more tractable (see Kemna and Vorst (1990)), the option payoffs are less effective from a hedging standpoint.

[17] Asian options are particularly useful when the underlying is thinly traded and subject to manipulation. The markets for commodities and some currencies qualify. They have lower premiums than standard European-style options and offer general protection in situations where regular cash flows need to be hedged. Occasionally, the average rate period applies for a short period at the end of the option's life. This period is referred to as the *Asian tail*.

Compute the value of an average rate call option with one year to expiration and an exercise price of $50. Assume that the average is computed using end-of-month asset prices, the asset's current price is $50, its income rate is 1%, and its volatility rate is 40%. The risk-free interest rate is 5%.

The steps of the Monte Carlo simulation, as applied to the valuation of average rate options, are pre-programmed in the OPTVAL function,

OV_APPROX_ASIAN_OPT_MC(*s, x, t, r, i, v, n, ntrial, cp, sx, ag*)

The first six parameters are already known. The parameter *n* is the number of observations used in computing the average. If the option's life is one year and *n* is set equal to 12, the average is computed based on monthly asset prices. The parameter *ntrial* is the number of simulation runs. The parameter *cp* is either "C" or "P," depending upon whether you are valuing a call or a put. The parameter *sx* is either "S" or "X," depending upon whether you are averaging the asset price to replace the asset price or the exercise price of the average rate option. Finally, the parameter *ag* is either "A" or "G," depending upon whether the average rate of the option is arithmetic or geometric. For the illustration at hand,

OV_APPROX_ASIAN_OPT_MC(50, 50, 1, 0.05, 0.01, 0.40, 12, 10000, "C", "S", "A") = 5.2812

In applying the OPTVAL function, you will not get exactly the same answer since the random drawings from the normal distribution will not be exactly the same. It is also worthwhile to note that if quarter observations are used in the computation of the average, the value of the option increases, that is,

OV_APPROX_ASIAN_OPT_MC (50, 50, 1, 0.05, 0.01, 0.40, 4, 10000, "C", "S", "A") = 5.9598

The reason is, of course, that variance of the average asset price and the option's value shrink as the number of observations increases. In the extreme case where the number of observations going into the computation of the average is one, the average rate option value is

OV_APPROX_ASIAN_OPT_MC (50, 50, 1, 0.05, 0.01, 0.40, 1, 10000, "C", "S", "A") = 8.7833

and should be identically equal to the BSM value of a European-style option,

OV_OPTION_VALUE(50, 50, 1, 0.05, 0.01, 0.40, "C", "E") = 8.7017

The small difference arises because Monte Carlo simulation is an approximation method.

Two Underlying Sources of Risk

The Monte Carlo simulation can be extended to handle multiple sources of risk. Consider a European-style put option on the minimum, for example. Since the option is written on the minimum of two risky assets, the option has two underlying sources of price risk. At expiration, the option holder receives

$$\max[X - \min(\tilde{S}_{1,T}, \tilde{S}_{2,T})]$$

If both asset prices follow geometric Brownian motions, the option can be valued analytically, as was noted earlier in the chapter. Nonetheless, we will value the option numerically using Monte Carlo simulation. In order to do so, we must explicitly account for the fact that movements in the asset prices are likely to be correlated with one another.

To handle the correlation between movements in asset prices, we make one small change to the Monte Carlo simulation procedure. First, we draw a unit normal random deviate for asset 1. Label it ε_1. Next, we draw a second unit normal random deviate for asset 2. Label it ε_2. Naturally ε_1 is uncorrelated with ε_2 since they are independent drawings. To induce correlation between the deviates, we apply the following transformation to the second deviate,

$$\varepsilon_2' = \rho\varepsilon_1 + \sqrt{1-\rho^2}\varepsilon_2 \tag{9.22}$$

where ρ is the correlation between the variables in the bivariate distribution. Note that this new variable ε_2' remains unit normal. Its mean is

$$E(\varepsilon_2') = \rho E(\varepsilon_1) + \sqrt{1-\rho^2}E(\varepsilon_2) = 0$$

and its variance is

$$\text{Var}(\varepsilon_2') = \rho^2\text{Var}(\varepsilon_1) + (1-\rho^2)\text{Var}(\varepsilon_2) = 1$$

Note also that the covariance (correlation) between ε_1 and ε_2' is

$$\begin{aligned}\text{Cov}(\varepsilon_1, \varepsilon_2') &= \text{Cov}(\varepsilon_1, \rho\varepsilon_1 + \sqrt{1-\rho^2}\varepsilon_2) \\ &= \rho\text{Var}(\varepsilon_1) + \text{Cov}(\varepsilon_1, \sqrt{1-\rho^2}\varepsilon_2) \\ &= \rho\end{aligned}$$

We then generate prices for asset 1 and asset 2 using ε_1 and ε_2'

$$S_{1,t+\Delta t} = S_{1,t}e^{(b_1-\sigma_1^2/2)\Delta t + \sigma_1\sqrt{\Delta t}\varepsilon_1} \tag{9.23}$$

and

$$S_{2,t+\Delta t} = S_{2,t}e^{(b_2-\sigma_2^2/2)\Delta t + \sigma_{21}\sqrt{\Delta t}\varepsilon_2'} \tag{9.24}$$

With the prices at each time step identified, we value the option.

ILLUSTRATION 9.6 Find value of European-style call on maximum of two risky assets.

Consider a call option that provides its holder the right to buy \$100,000 worth of the S&P 500 index portfolio at an exercise price of \$1000 or \$100,000 worth of a particular T-bond at an exercise price of \$100, whichever is worth more at the end of three months. The S&P 500 index is currently priced at \$1075, pays dividends at a rate of 1% annually and has a return volatility of 18%. The T-bond is currently priced at \$105, pays a cou-

pon yield of 6% and has a return volatility of 8%. The correlation between the rates of return of the S&P 500 and the T-bond is 0.5. The risk-free rate of interest is 3%.

Before applying the Monte Carlo simulation technique, it is important to recognize that there are two exercise prices in this problem: $1,000 for the S&P index portfolio and $100 for the T-bond. What this implies is that we can buy 100,000/1,000 = 100 units of the index portfolio or 100,000/100 = 1,000 units of T-bonds at the end of three months, depending on which is worth more. At this juncture, we must decide whether to value the call option on the maximum in units of the S&P 500 index portfolio, in which case we multiply the current T-bond price and its exercise price by 10 and then multiply the computed option price by 100, or to value the option in units of the T-bond, in which case we divide the current S&P 500 price and the option's S&P 500 exercise price by 10 and then multiply the computed option price by 1,000.[18] We choose to work in units of the S&P 500 index portfolio, so we adjust the T-bond prices: the current T-bond price is assumed to be 10,500 and the T-bond exercise price is 1,000. With the units of the two underlying assets comparable, we apply the OPTVAL function,

OV_APPROX_MAXMIN_OPT_MC(s1,s2,x,t,r,i1,i2,v1,v2,rho,n,ntrial,cp,mm)

to find

OV_APPROX_MAXMIN_OPT_MC(1100,980,1000,0.25,0.03,0.01,0.06,0.18,0.08,0.5,10, 10000,"C", "X") = 98.671

The computed option value is 98.671 per $1,000 or $98.671 × 1,000 = $9,867.10 in total.

Mean Reversion

Another advantage in using Monte Carlo simulation is that other processes for asset price movements can be introduced seamlessly. While an assumption geometric Brownian motion may be sensible for movements in the price of underlying assets such as stocks and stock indexes which tend to grow through time, the prices of assets such as gold and oil as well as interest rates on bonds tend to revert back to mean levels.[19] A simple mean reversion process is

$$dS = \kappa(\theta - S)dt + \sigma dz \tag{9.25}$$

where κ is the continuous-time speed of mean reversion or pull rate, θ is the mean reversion level, and σ is the continuous-time standard deviation of the price changes.[20] Under this assumption, discrete movements in asset price through time are described by

[18] These types of adjustments can be made freely because the option price is linearly homogeneous in both the asset price and the exercise price. See Merton (1973).

[19] Schwartz and Smith (1999), for example, model the short-term movements of commodity prices as a mean-reverting process and model movements in the long-term equilibrium price as a Brownian motion. asicek (1977) models movements in short-term interest rate as a mean-reverting process.

[20] This process is commonly referred to as an Ornstein-Uhlenbeck process or, alternatively, a Gauss-Markov process.

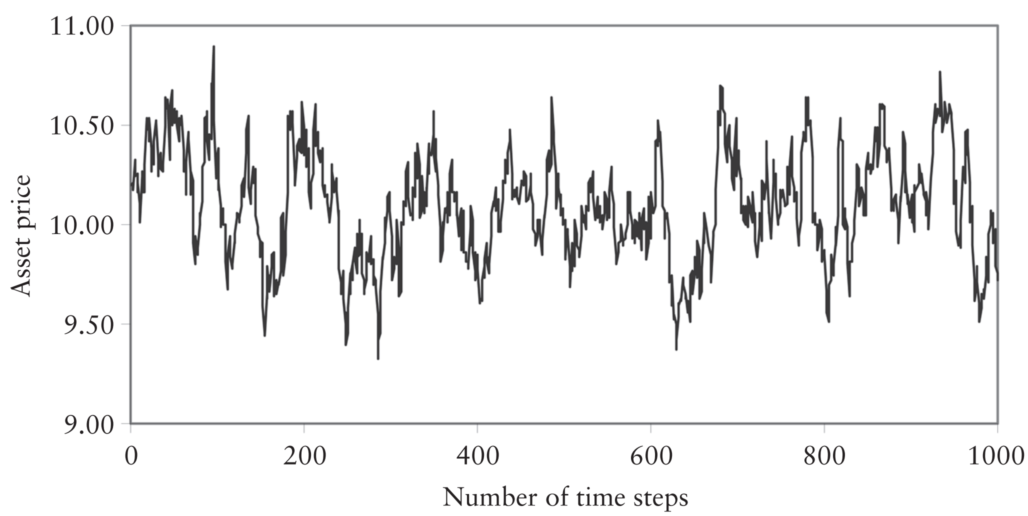
$$S_{t+\Delta t} - S_t = \theta k - k S_t + \sigma_{\Delta t}\varepsilon \tag{9.26}$$

where $k = 1 - e^{-\kappa\Delta t}$ is the discrete-time reversion rate over the interval Δt,

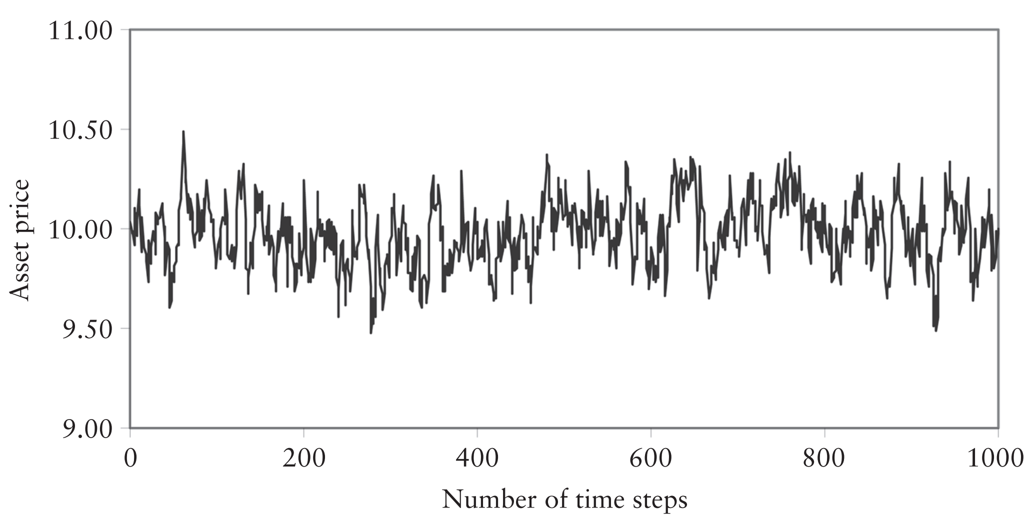
$$\sigma_{\Delta t} = \sqrt{\frac{1 - e^{-2\kappa\Delta t}}{2\kappa\Delta t}}\sigma$$

is the discrete-time volatility, and where ε is a drawing from a unit normal distribution. Figure 9.15, Panels A and B, show simulated price movements of an

FIGURE 9.15 Simulated asset price path for mean reversion process.
Panel A: Mean reversion rate of 0.05.



Panel B: Mean reversion rate of 0.20.



asset that follows a mean reversion process. The processes include 1,000 time steps, with each time step $\Delta t = 1$. The mean reversion level is 10, and the continuous-time volatility rate is 0.10 (i.e., discrete-time volatility rate of 0.09755). Panel A has a continuous-time reversion rate of 0.05 (i.e., a discrete-time reversion rate of 0.04877), while Panel B has a reversion rate of 0.20 (i.e., a discrete-time reversion rate of 0.018127). As the figures show, the higher the rate of mean reversion, the quicker price is pulled back toward the mean, and the less the variation in price. To value options on an asset whose price is mean-reverting, we simply generate the asset price path using (9.26) rather than (9.20). All other steps in the Monte Carlo valuation procedure are the same.[21]

QUADRATIC APPROXIMATION

The quadratic approximation, developed by MacMillan (1986) and Barone-Adesi and Whaley (1987),[22] is based on the simple notion that an American-style option can be thought of as the sum of an otherwise similar European-style option and an early exercise premium, that is,

$$C = c + \varepsilon_C \tag{9.27}$$

and

$$P = p + \varepsilon_P \tag{9.28}$$

where ε_C and ε_P are the early exercise premiums on the American-style call and put, respectively. The virtue in doing so is that the European-style options have analytical valuation equations. In the last chapter, the European-style option valuation formulas were presented as

$$c = Se^{-iT}N(d_1) - Xe^{-rT}N(d_2) \tag{9.29}$$

and

$$p = Xe^{-rT}N(-d_2) - Se^{-iT}N(-d_1) \tag{9.30}$$

where

$$d_1 = \frac{\ln(Se^{-iT}/Xe^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}}, \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

[21] In order to apply the risk-neutral option valuation principles, we must adjust the rate of drift by the market price of risk—a discussion which we defer to a later chapter.
[22] The quadratic approximation is also applied to futures options in Whaley (1986).

The term $N(d)$ is the cumulative normal density function, as defined in the last chapter.

Under the quadratic approximation, the value of an American-style call on an asset with a constant, continuous carry rate is

$$C = \begin{cases} c + A_2(S/S^*)^{q_2} & \text{if } S < S^* \\ S - X & \text{if } S \geq S^* \end{cases} \tag{9.31}$$

where

$$A_2 = \frac{S^*\{1 - e^{-iT}N[d_1(S^*)]\}}{q_2},\ d_1(S) = \frac{\ln(Se^{-iT}/Xe^{-rT}) + 0.5\sigma^2 T}{\sigma\sqrt{T}},$$

$$q_2 = \frac{1 - n + \sqrt{(n-1)^2 + 4k}}{2},\ n = \frac{2(r-i)}{\sigma^2},\ k = \frac{2r}{\sigma^2(1 - e^{-rT})}$$

c is the value of the corresponding European-style call option using (9.29), and S^* is the critical asset price above which the American-style call should be exercised immediately and is the solution to

$$S^* - X = c(S^*) + \{1 - e^{-iT}N[d_1(S^*)]\}S^*/q_2 \tag{9.32}$$

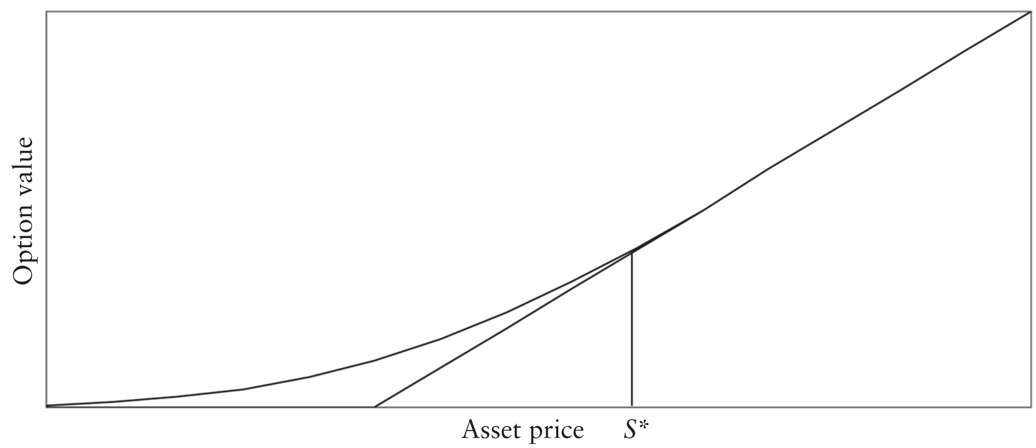
Figure 9.16, Panels A and B, provide some intuition for how the quadratic approximation works. Before valuing the call option using (9.31), it is necessary to identify the critical asset price above which the call will be exercised immediately. This is done using (9.32). The critical price, S^*, is that unique asset price at which (9.32) holds as an equality and does not depend on the current asset price. Panel A shows where this price lies. Below S^*, the call is valued using the first line on the right-hand side of (9.31). Above S^*, the call value is simply the difference between the asset price and the exercise price (i.e., the second line on the right-hand side of (9.31)). With S^* in hand, we can then generate call option values over a range of asset prices. This is done in Panel B of Figure 9.16. Note that the early exercise premium—the difference between the American-style and the European-style call option values—grows large as the asset price rises.

The quadratic approximation for an American-style put on an asset with a constant, continuous carry rate is

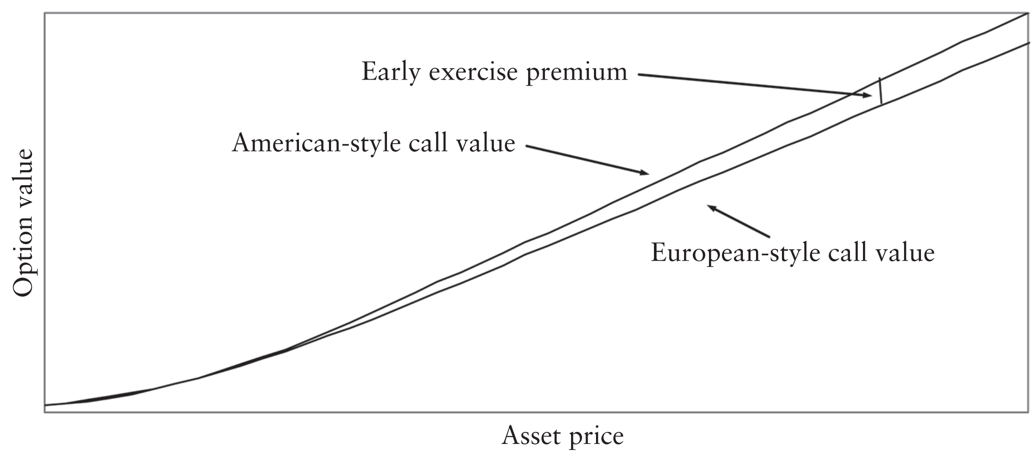
$$P = \begin{cases} p + A_1(S/S^{**})^{q_1} & \text{if } S > S^{**} \\ X - S & \text{if } S \leq S^{**} \end{cases} \tag{9.33}$$

where

FIGURE 9.16 Illustration of the components of the quadratic approximation method.
Panel A: Determination of critical asset price, S^*.



Panel B: Early exercise premium.



$$A_1 = -\frac{S^{**}\{1 - e^{-iT}N[-d_1(S^{**})]\}}{q_1}$$

$$q_1 = \frac{1 - n - \sqrt{(n-1)^2 + 4k}}{2}$$

p is the value of the corresponding European-style put option using (9.32), and S^{**} is the critical asset price below which the American-style put should be exercised immediately and is the solution to

$$X - S^{**} = p(S^{**}) - \{1 - e^{-iT}N[-d_1(S^{**})]\}S^{**}/q_1 \tag{9.34}$$

ILLUSTRATION 9.7 Compute value of American-style option using quadratic method.

Compute the value of a two-year American-style put option with an exercise price of 55. Use the quadratic approximation. Assume the underlying asset is a foreign currency whose current price is 50 and whose volatility rate is 20%. Assume the domestic rate of interest is 5%, and the foreign rate of interest, 2%. Use the quadratic approximation, and show intermediate computations.

To begin, use the domestic and foreign interest rates to identify the expected rate of price appreciation of the currency, that is,

$$b = r_d - r_f = 0.05 - 0.02 = 0.03$$

Next compute the values of n and k. Using the problem parameters, these are

$$n = \frac{2b}{\sigma^2} = \frac{2(0.03)}{0.20^2} = 1.5$$

and

$$k = \frac{2r}{\sigma^2(1-e^{-rT})} = \frac{2(0.05)}{0.20^2(1-e^{-0.05(2)})} = 26.2708$$

With the values of n and k, can compute the value of q_1, that is,

$$q_1 = \frac{1-1.5-\sqrt{(1.5-1)^2+4(26.2708)}}{2} = -5.3816$$

Now the critical asset price below which you would exercise the American-style put immediately, S^{**}, must be identified by solving

$$55 - S^{**} = (p(S^{**}) + \{1-e^{-(0.03-0.05)2}N[-d_1(S^{**})]\}S^{**})/5.3816$$

The value S^{**} of that satisfies the equation is 41.1776. This means the value of A_1 is

$$A_1 = \frac{41.1776\{1-e^{-(0.03-0.05)2}N[-d_1(41.1776)]\}}{5.3816} = 2.1490$$

With all of the parameters identified, put valuation becomes a matter of applying

$$P = \begin{cases} p + 2.1490(S/41.1776)^{-5.3816} & \text{if } S > 41.1776 \\ 55 - S & \text{if } S \leq 41.1776 \end{cases}$$

where p is the value of a European-style put with the same parameters. At the currency of 50, the value of the European-style put is 6.41, and the value of the American-style put is 7.16. This implies that the early exercise premium has a value of approximately 75 cents.

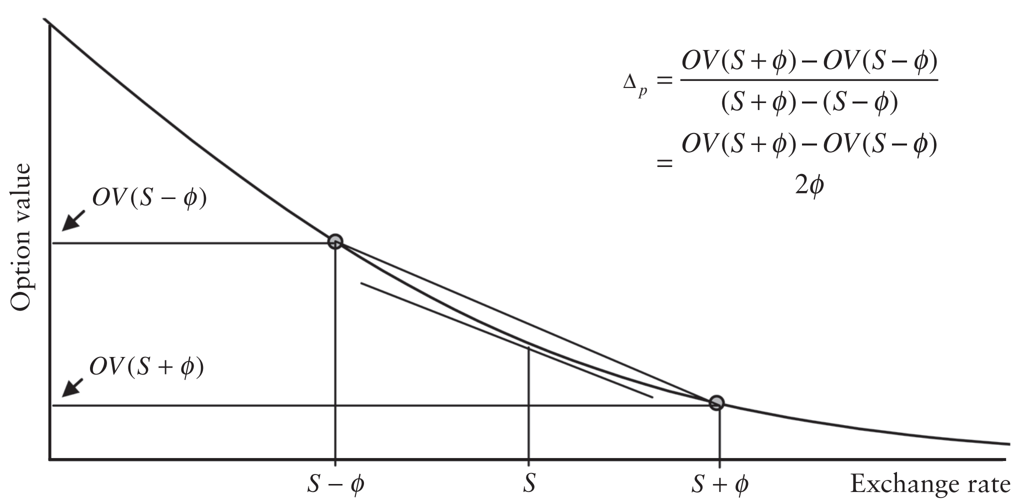
The quadratic approximation technique is not as flexible as lattice-based methods for valuing options with nonstandard features. For standard options on assets with a constant continuous carry rate (e.g., foreign currency options, stock index options, futures options), however, the quadratic approximation is faster and more accurate than competing methods.

MEASURING RISK NUMERICALLY

Just as the options studied in this chapter must valued numerically, the risk characteristics of these options must be computed numerically. The procedure is straightforward. Recall that the delta of an option is the change in option value with respect to a change in asset price. To obtain the delta of an option numerically, perturb the current asset price S by a small amount ϕ in either direction, that is, $S + \phi$ and $S - \phi$, and value the option at each asset price, $OV(S + \phi)$ and $\mathrm{OV}(S - \phi)$. Figure 9.17 illustrates. The valuation function $OV(.)$ can be any of the valuation methodologies discussed in this chapter. In the figure, the quadratic approximation was used to generate the values of an American-style FX put option for different levels of the exchange rate. Ideally, we would like to know the slope of the OV function at the current exchange rate S. We cannot do so by taking the partial derivative of OV with respect to S because we do not have an analytical expression for OV. To approximate the delta, therefore, we take the ratio of the difference between the computed option values to the difference between the perturbed asset prices, that is,

$$\Delta_p = \frac{OV(S+\phi) - OV(S-\phi)}{(S+\phi) - (S-\phi)}$$

FIGURE 9.17 Numerical approximation for the delta of a FX put option.



To assess the accuracy of this numerical procedure, consider the two-year put option that we have used as an illustration throughout the chapter, but assume, for the moment, that put is European-style. The *analytical* option value is 6.406 and, using the formula from the last chapter, its *analytical* delta is –0.474. Now suppose that there is no formula for computing the delta, and, instead, it must be computed using the numerically. First, perturb the asset price up by, say, 25 cents to a level of 50.25, and compute the put value. Using the BSM formula, the put value is 6.288. Next perturb the asset price downward by the same amount. The put option value at an asset price of 49.75 is 6.525. The numerical delta for this put is therefore

$$\Delta_p = \frac{6.288 - 6.525}{(50 + 0.25) - (50 - 0.25)} = \frac{-0.237}{0.50} = -0.474$$

Not surprisingly, the *analytical* delta and the *numerical* delta values are very close. At three decimal places the difference is not apparent. At six decimal places, the analytical delta is –0.474039 and the numerical delta is –0.474038.

In general, all of the Greeks for options can be measured using the expression

$$\text{Greek}_k = \frac{OV(k + \phi) - OV(k - \phi)}{2\phi} \tag{9.35}$$

where OV represents one of the numerical valuation methods that we described earlier in the chapter (e.g., the binomial method, the trinomial method, Monte Carlo simulation, and the quadratic approximation), k is the option determinant of interest (e.g., S for delta risk, σ for vega risk, and so on), and ϕ is a small positive constant selected by the user. The gamma, that is, the change in the delta with respect to a change in the asset price, may be computed using

$$\text{Gamma} = \frac{OV(S + \phi) - 2 \times OV(S + \phi) + OV(S - \phi)}{\phi^2} \tag{9.36}$$

It is also worth noting that, if the delta and theta of the option have already been computed, the gamma can be solved for analytically.[23] Recall that the Black-Scholes/Merton partial differential equation for valuing options (i.e., equation (7D.6) from Appendix 7D in Chapter 7) may be written

$$-\theta + bS\Delta + \frac{1}{2}\sigma^2 S^2 \gamma = rOV \tag{9.37}$$

where OV is the option value. Rearranging to isolate gamma,

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[23] This idea was first suggested by Carr (2001).

$$\gamma = \frac{rOV + \theta - bS\Delta}{0.5\sigma^2 S^2} \qquad (9.38)$$

The formulas (9.35) and (9.36) provide the means for calculating the Greeks numerically and can be applied to obtain any risk measures detailed in Table 9.4. The numerical values of the Greeks for the American-style put illustration maintained throughout the chapter are reported in Table 9.5. The quadratic approximation is used to value the put. The values reported in the table are identical to those that would be obtained by using the OV_OPTION_VALUE function from the OPTVAL Library. Note that condition (9.38) is satisfied.

SUMMARY

Numerical methods are an indispensable tool in option valuation and risk measurement. The reasons are that numerical methods: (1) are required in instances where the option valuation problem is intractable from a mathematical standpoint (i.e., American-style options); (2) are more convenient than analytical methods in situations where the option valuation problem has large numbers of

TABLE 9.4 Details of formula use for evaluating option risk measures numerically.

Change in option value with respect to a change in:	Determinant, *k*	Greek$_y$	Symbol
Asset price	S	Delta	Δ
Interest rate	r	Rho$_r$	ρ^r
Income rate	i	Rho$_i$	ρ^i
Volatility rate	σ	Vega	Vega
Time to expiration	T	Theta	θ
Change in delta with respect to a change in:			
Asset price	S	Gamma	γ

TABLE 9.5 Summary of risk characteristics of the American-style put written on a currency evaluated numerically using the quadratic approximation.

Partial Derivative with Respect to	Greek Symbol	Perturbation Amount, ϕ	Numerical Value of Greek
S	Δ	0.25	–0.555
Δ	γ	0.25	0.037
r	ρ^r	0.05%	–35.384
i	ρ^i	0.05%	30.416
σ	Vega	0.05%	25.736
T	θ	0.01	0.706

contingencies (e.g., accrual options); and (3) can be extremely accurate if applied properly. This chapter examines three different numerical methods—the binomial method, the trinomial, method and Monte Carlo simulation. All three methods involve replacing the BSM assumption that the underlying asset price has continuous geometric Brownian diffusion with an assumption that underlying asset price jumps over small discrete time intervals during the option's life. The *binomial method,* for example, assumes that the asset price moves to one of two levels over the next increment in time. The size of the move and its likelihood are chosen in a manner so as to be consistent with the log-normal asset price distribution. In a similar fashion, the *trinomial method* allows the asset price to move to one of three levels over the next increment in time, and *Monte Carlo simulation* uses a discretized version of geometric Brownian motion to enumerate every possible path that the asset's price may take over the life of the option. Each of these three numerical methods is illustrated using a variety of derivatives contracts including standard American-style options, spread options, Asian-style options, and options on the minimum and maximum. For valuing standard American-style options, the *quadratic approximation method* is also discussed. It addresses the value of early exercise by modifying the BSM partial differential equation.

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