CHAPTER 20

Valuing Interest Rate Products Numerically

Valuing interest rate derivatives written on short-term bonds is trickier than valuing derivatives on other types of assets for two reasons. First, for an asset such as a stock, a currency or a commodity, price can roam freely through time without constraint. For a fixed income security, however, price is often forced to take a particular level when the security matures. A T-bill, for example, has a value of 100 when it matures, and a T-note has a terminal payment equal to its final coupon interest payment plus the par value. Second, in the fixed income markets, there is often a wide range of securities available on the *same* underlying source of uncertainty. The U.S. Treasury, for example, has T-bills, T-notes and T-bonds with a wide range of maturities. In modeling interest rate dynamics, care must be taken to ensure that all of these securities are simultaneously valued at levels consistent with observed market prices.

The purpose of this chapter is modest—to develop a binomial procedure for valuing interest rate derivative contracts where the short-term interest rate ("short rate") is the single underlying source of interest rate uncertainty. To begin, we discuss a number of constant-parameter short rate processes to lay a foundation for interest rate behavior. While these models are often useful in developing economic intuition regarding interest rate behavior, they produce zero-coupon bond values that are different from the observed market prices, seemingly giving rise to arbitrage opportunities. Consequently, we next turn to no-arbitrage pricing models. These models adjust the parameters of the interest rate process in a manner that produces bond (and interest rate derivatives contract) values equal to observed prices. With the mechanics of no-arbitrage pricing in hand, we then turn to valuing zero-coupon and coupon-bearing bonds, callable bonds, putable bonds, and bond options. Be forewarned, however. While the valuation framework provided in this chapter is intuitive and commonly-applied in practice, it only begins to scratch the surface of the literature focused on no-arbitrage interest rate models. This literature is deep in multifactor theoretical models of interest rate movements and numerical procedures for calibrating the interest rate models and valuing interest rate derivatives.

CONSTANT-PARAMETER MODELS

In the Black-Scholes (1973)/Merton (1973) model developed in Chapter 5, the price of an asset was assumed to follow the geometric Brownian motion (i.e., equation (5.4)), that is,

$$dS = \alpha S dt + \sigma S dz \tag{20.1}$$

This assumption implies that, over the next infinitesimally small interval of time *dt*, the change in asset price, *dS*, equals an expected price increment (i.e., the product of the instantaneous expected rate of change in asset price, α, times the current asset price, *S*, times the length of the interval) plus a random increment proportional to the instantaneous standard deviation of the rate of change in asset price, σ, times the asset price. Note that, in the assumed process (20.1), the parameters α and σ are constants (i.e., do not vary through time or with the level of asset price). In the first part of this section, we develop economic intuition regarding plausible interest rate processes by examining four constant-parameter interest rate processes. In the second part, we show why constant-parameter, short rate models are seldom used in practice.

Constant-Parameter, Short Rate Processes

The simplest constant-parameter, short rate process that we consider is the arithmetic Brownian motion assumption,

$$dr = a dt + \sigma dz \tag{20.2}$$

where *dr* is the instantaneous change in the short rate, *a* is its instantaneous mean, and σ is its instantaneous standard deviation. The assumption (20.2) says that the short-rate change over the next increment in time, Δt, is normally distributed with mean $r + a\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$.[1] If $a > 0$, the short rate is expected to climb through time, and, if $a < 0$, it is expected to fall. The size of the random change in the rate increases proportionally with $\sigma\sqrt{\Delta t}$.

In terms of describing interest rate dynamics, the process (20.2) has a number of weaknesses. First, the process does nothing to guard against the possibility of the short rate becoming negative. In particular, if $a < 0$, the short rate must eventually become negative. Similarly, the short rate can become negative in the stochastic component of the short-rate movement (i.e., the second term on the right-hand side of (20.2)) has a large negative value. Naturally, in a rationally functioning marketplace, negative interest rates will not arise. In such an environment, individuals would prefer to put cash in their mattresses than hold Treasury bills.

A second weakness of (20.2) is that, if $a > 0$, the short rate is expected to rise without limit. While this assumption may be plausible for asset prices,

[1] For clarity of exposition, think of the short rate *r* as being the continuously compounded interest rate on a one-year U.S. T-bill and the time increment Δt as being equal to one year.

casual empirical observation suggests that interest rates tend to revert toward some long-run mean level through time.[2] This stands to reason from an economic standpoint. When interest rates are high, the demand for borrowed funds subsides, causing interest rates to fall. Conversely, when interest rates are low, the demand for borrowed funds rises, causing interest rates to rise. A third weakness of (20.2) is that the volatility rate is the same, independent of whether interest rates are high or low. From an empirical standpoint, the volatility of interest rates tends to rise with as the level of interest rates rises and falls as the level of interest rates falls.

The next constant-parameter, short-rate process that we consider is the geometric Brownian motion assumption,

$$dr = ardt + \sigma rdz \tag{20.3}$$

introduced by Rendleman and Bartter (1980). In (20.3), a is the instantaneous expected rate of change in the short rate, and σ is its instantaneous standard deviation. Note that this specification is identical to the BSM assumption (20.1), that is, Rendleman and Bartter assume that the short rate behaves as if it were an asset price. The process (20.3) circumvents two of the weaknesses associated with (20.2). First, with (20.3), interest rates cannot become negative. One reason is that the expected short rate at the end of the next increment in time is $re^{a\Delta t}$. Even if $a < 0$, the expected short rate remains positive. Another is that the stochastic component of interest rate movements (i.e., the second term on the right-hand side of (20.3)) approaches zero as interest rates fall. Second, the process (20.3) captures the empirical phenomenon that the volatility of interest rates changes (σr in this case) increases with the level of interest rates. The one weakness that (20.3) does not circumvent, however, is that if $a > 0$ the short rate is expected to rise without limit. The process fails to account for the empirical fact (and economic prediction) that interest rates are mean-reverting.

Next is the short-rate process derived by Vasicek (1977),

$$dr = a(b - r)dt + \sigma dz \tag{20.4}$$

where the parameters a, b and σ are constants. Like (20.2) and (20.3), the change in the short rate has an expected and a random component. Unlike the first terms on the right-hand sides of (20.2) and (20.3) where the short rate is expected to drift upward or downward, however, the first term on the right-hand side of the Vasicek model (20.4) captures mean reversion in the short rate. The long-run mean level of the short rate is b, so, if the current short rate r is less than b, the short rate is pulled upward, and, if the current short rate is above b, it is pulled downward (assuming, of course, that a is positive). The parameter a is called the rate of pull or, simply, pull rate. If the pull rate is 0.5 and the current short rate r is 1% below the long-run mean b, we expect that the short rate will increase by 0.5% over the next increment in time. If $a = 0$, the short rate follows arithmetic Brownian motion with a zero mean (i.e., a random walk). Where $a = 1$, the short rate is expected to immediately return to its long-

[2] Recall that we first discussed mean reversion in Chapter 9.

term mean. The last term on the right-hand side accounts for random movements in the short rate. Like (20.2), the random changes in the short rate in (20.4) are a normally distributed and independent of the level of the short rate. This means, like (20.2), the short rate in (20.4) has the prospect of becoming negative and does not account for the fact that the volatility of interest rates changes tends to increase with the level of interest rates and vice versa.

The fourth and final constant-parameter, short rate process that we consider was derived by Cox, Ingersoll, and Ross (1977). The CIR model is specified as

$$dr = a(b - r)dt + \sigma\sqrt{r}dz \tag{20.5}$$

The first term on the right-hand side (20.5) is the mean reversion component introduced by Vasicek. Unlike the Vasicek model, however, the instantaneous standard deviation by the factor $\sqrt{r}$. This overcomes the remaining two deficiencies of the Vasicek model. Specifically, with the random component of the interest rate change defined as $\sigma\sqrt{r}$, (1) the volatility of interest rate movements is directly related to the level of interest rates; and (2) negative interest rates are not possible (i.e., where the short rate falls to zero, the second term on the right-hand side approaches zero, and the short rate is guaranteed to move upward).

Applying Constant-Parameter Models

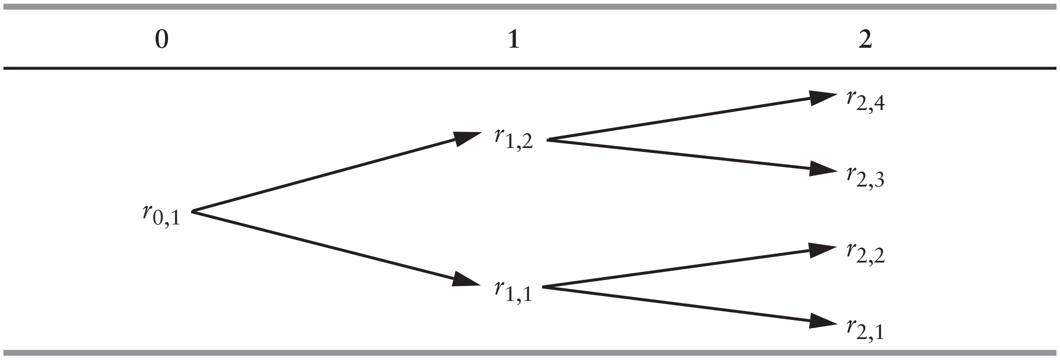
All of the constant-parameter models described above can be implemented for valuing bonds and interest rate derivatives. None of them will produce values that are completely consistent with prices observed in the marketplace, however. The reason is that the parameters of the model are constant through time. To see this, consider applying the Vasicek model to value zero-coupon bonds. We begin by approximating (20.4) using the binomial distribution,

$$r_{t+\Delta t} - r_t = \begin{cases} a(b - r_t)dt + \sigma\sqrt{r}dz & \text{with probability} = 1/2 \\ a(b - r_t)dt - \sigma\sqrt{r}dz & \text{with probability} = 1/2 \end{cases} \tag{20.6}$$

Note that, by defining the short-rate movements as (20.6), the vertical distance between the two nodes emanating from r_t equals $2\sigma\sqrt{\Delta t}$.

One disadvantage of using the binomial method to approximate short-rate movements within the Vasicek model is that the binomial lattice does not recombine. To see this, recall the lattice notation from Chapter 9. Specifically, let $r_{i,j}$ be the short rate at time i and vertical node j, where $j = 1$ is the lowest node at time i. Figure 20.1 contains a two-period, short-rate lattice. Note that at time 2, there are four nodes rather than three since, in general, $r_{2,3} \neq r_{2,2}$. The only instance in which the nodes will recombine (i.e., $r_{2,3} = r_{2,2}$) is where $a = 0$, in which case the short rate follows a simple random walk. The fact that the binomial lattice does not recombine does not mean that the binomial method cannot be used in this context. It only means that the computational exercise is more tedious. With a recombining lattice, the number of possible interest rate nodes is $n + 1$. With a nonrecombining lattice, the number of nodes is 2^n. Where

FIGURE 20.1 Two-period lattice for Vasicek model.



where the nodes at time 1 are

$$r_{1,2} = r_{0,1} + a(b - r_{0,1})\Delta t + \sigma\sqrt{\Delta t}$$

and

$$r_{1,1} = r_{0,1} + a(b - r_{0,1})\Delta t - \sigma\sqrt{\Delta t}$$

and the nodes at time 2 are

$$r_{2,4} = r_{1,2} + a(b - r_{1,2})\Delta t + \sigma\sqrt{\Delta t}$$
$$r_{2,3} = r_{1,2} + a(b - r_{1,2})\Delta t - \sigma\sqrt{\Delta t}$$
$$r_{2,2} = r_{1,1} + a(b - r_{1,1})\Delta t + \sigma\sqrt{\Delta t}$$
$$r_{2,1} = r_{1,1} + a(b - r_{1,1})\Delta t - \sigma\sqrt{\Delta t}$$

Note that, in general, $r_{2,3} \neq r_{2,2}$.

the number of time steps is 10 (i.e., $n = 10$), the number of nodes is 101 for a recombining lattice and 1,024 for a nonrecombining lattice.

Now, let us consider valuing zero-coupon or discount bonds using (20.6). Assume that the zero-coupon yield curve is given by

$$r_i = 0.10 - 0.05e^{0.18(T_i - 1)}$$

where r_i is the continuously compounded, zero-coupon yield rate, and T_i is its time to maturity measured in years. Also assume that we have obtained a history of one-year short rates and have estimated the parameters of the Vasicek model to be $a = 0.5$, $b = 0.06$, and $\sigma = 0.01$, where b and σ are annualized rates.[3] Now, let us compute the one-year short rates using Vasicek's mean-reverting process, and then value one-year, two-year, and three-year discount bonds.

[3] Recall that in Chapter 9 we showed how to estimate the parameters of a mean-reverting process using regression analysis.

FIGURE 20.2 Two-period lattice for Vasicek model assuming the current short rate is 5%, the pull rate is 0.5, the long-run average short rate is 6%, and the volatility rate is 1%. ($r = 0.05$, $a = 0.5$, $b = 0.06$, $\sigma = 0.01$).

0	1	2
		7.250%
	6.500%	
		5.250%
5.000%		
		6.250%
	4.500%	
		4.250%

Figure 20.2 shows the evolution of the short rate under the assumed parameter values. With the current one-year short rate at 5%, the possible one-year short rates in one year are

$$r_{1,2} = 0.05 + 0.5(0.06 - 0.05)1 + 0.01\sqrt{1} = 0.065$$

and

$$r_{1,1} = 0.05 + 0.5(0.06 - 0.05)1 - 0.01\sqrt{1} = 0.045$$

The expected one-year short rates in two years are

$$r_{2,4} = 0.065 + 0.5(0.06 - 0.065)1 + 0.01\sqrt{1} = 0.0725$$

$$r_{2,3} = 0.065 + 0.5(0.06 - 0.065)1 - 0.01\sqrt{1} = 0.0525$$

$$r_{2,2} = 0.045 + 0.5(0.06 - 0.045)1 + 0.01\sqrt{1} = 0.0625$$

and

$$r_{2,1} = 0.045 + 0.5(0.06 - 0.045)1 - 0.01\sqrt{1} = 0.0425$$

Note that the lattice in Figure 20.2 shows the mechanics of short-rate mean reversion at work. Each year, the one-year short-rate jumps up or down by 1% due to the volatility component (i.e., $\pm 0.01\sqrt{1}$). Viewed in isolation, this means than standing at $r_{1,2} = 6.5\%$, the one-year short rate will jump to 7.5% or 5.5% with equal probability. But, because the one-year spot rate is above the long-run mean level of 6%, the subsequent one-year spot rates are pulled toward the long-run mean by an amount equal to 0.5(0.06 – 0.065)1 = 0.0025 or 0.25%. Thus, the nodes $r_{2,4}$ and $r_{2,3}$ are 7.25% and 5.25%, respectively.

Based on the evolution of one-year spot rates displayed in Figure 20.2, we can now compute the values of one-year, two-year, and three-year discount bonds. A one-year discount bond pays 1 in one-year. The one-year short rate is known to be 5%. The value of a one-year discount bond is therefore

$$DBV_1 = e^{-0.05(1)} = 0.95123$$

A two-year, zero-coupon bond pays 1 in year 2. According to the interest rate lattice in Figure 20.2, the evolution of the short rate is (1) 5% over the first year and 6.5% over the second or (2) 5% over the first year and 4.5% over the second, with equal probability. The value of a two-year discount bond is therefore

$$DBV_2 = 0.5e^{-0.05(1)}e^{-0.065(1)} + 0.5e^{-0.05(1)}e^{-0.045(1)} = 0.90037$$

Finally, a three-year discount bond pays 1 in three years. Again, the interest rate lattice in Figure 20.2 describes the possible paths for the short-rate evolution. Four paths are possible, each with equal probability: (1) 5%, 6.5%, and 7.25%, (2) 5%, 6.5%, and 5.25%, (3) 5%, 4.5%, and 6.25%, and (4) 5%, 4.5%, and 4.25%. The value of a three-year discount bond is

$$\begin{aligned} DBV_3 &= 0.25[e^{-0.05(1)}e^{-0.065(1)}e^{-0.0725(1)}] + 0.25[e^{-0.05(1)}e^{-0.065(1)}e^{-0.0525(1)}] \\ &\quad + 0.25[e^{-0.05(1)}e^{-0.045(1)}e^{-0.0625(1)}] + 0.25[e^{-0.05(1)}e^{-0.045(1)}e^{-0.0425(1)}] \\ &= 0.85015 \end{aligned}$$

Now, with the Vasicek model discount bond values in hand, recall that at the outset we assumed the zero-coupon yield curve was given by the relation $R_t = 0.10 - 0.05e^{0.18(t-1)}$. Such a yield curve implies that the zero-coupon bond prices at the outset are

Years to Maturity	Spot Rate	Discount Bond Price
1	5.000%	95.123
2	5.824%	89.005
3	6.512%	82.255

The one-year discount bond price matches its theoretical value because in applying the Vasicek model we assumed that the one-year short rate was 5%. The two-year and three-year discount bond prices do not match their theoretical values (0.89005 versus 0.90037 and 0.82255 versus 0.85015, respectively), however. The reasons for these apparent arbitrage opportunities are twofold. First, we used historical estimates of the parameters a, b, and σ, and, while assuming past parameters are reasonable predictions for the future, they may not be. Second, the Vasicek model assumes that the parameters a, b, and σ are constant through time. Such an assumption will give rise to apparent arbitrage opportunities because the interest rate dynamics modeled by (20.6) are not rich enough to describe the current term structure of zero-coupon interest rates.

FIGURE 20.3 Discount bonds values based on Vasicek model assuming the current short rate is 5%, the pull rate is 0.5, the long-run average short rate is 6%, and the volatility rate is 1%. ($r = 0.05$, $a = 0.5$, $b = 0.06$, $\sigma = 0.01$).

One-year discount bond value lattice:

0	1
	1
0.95123	
	1

Two-year discount bond value lattice:

0	1	2
	0.93707	1
0.90037		
	0.95600	1

Three-year discount bond value lattice:

0	1	2	3
		0.93007	1
	0.880334		
		0.94885	1
0.85015			
		0.93941	1
	0.90715		
		0.95839	1

One possible remedy to this problem is to *calibrate* the short-rate parameters using market prices.[4] More specifically, if we equate the model values of the discount bonds to their observed prices, we can infer the parameters, a, b, and σ. In the current illustration, we have two mismatched prices, so we can infer only two of the three model parameters. Suppose that we are willing to accept the fact that $\sigma = 0.01$. We can now solve for the parameters a and b by insisting that the two-year and three-year discount bond values equal their market prices. The parameter values of $a = 0.2440$ and $b = 0.1177$ will make the discount bond values equal their market prices,[5] as shown in Figure 20.4. The apparent arbitrage opportunities have disappeared, however, one is left with the uncomfortable situation that parameter values may not be reasonable from an economic standpoint. Such is the tradeoff created by applying no-arbitrage pricing models.

[4] We used the *calibration* process in earlier chapters when we computed implied standard deviations from option prices.
[5] Solving for the parameters a and b can be accomplished using the Microsoft Excel function, SOLVER.

FIGURE 20.4 Discount bonds values based on Vasicek model assuming the current short rate is 5%, the pull rate is 0.2440, the long-run average short rate is 11.77%, and the volatility rate is 1%. ($r = 0.05$, $a = 0.2440$, $b = 0.1177$, $\sigma = 0.01$).

One-year discount bond value lattice:

0	1
	1
0.95123	
	1

Two-year discount bond value lattice:

0	1	2
	0.92633	1
0.89005		
	0.94504	1

Three-year discount bond value lattice:

0	1	2	3
		0.90794	1
	0.84954		
		0.92628	1
0.82255			
		0.92177	1
	0.87991		
		0.94039	1

To emphasize the issue about the plausibility of the parameter estimates, we can extend the illustration to include four discount bond prices. With three mismatched prices, we can infer all three parameters of the Vasicek model. The no-arbitrage parameter values will be $a = 0.2494$, $b = 0.1161$, and $\sigma = -0.00009$. Although all discount bond values now match observed market prices, we are in the unpalatable position of explaining why the estimate of the standard deviation parameter is negative. Clearly, we have reached the limits of this constant-parameter model. Beyond four discount bond prices, it is impossible for the Vasicek model to be used within a no-arbitrage framework. Arbitrage opportunities will appear. The assumed stochastic process is simple not rich enough to capture interest rate movements.

NO-ARBITRAGE MODELS OF INTEREST RATES

As we have just shown, the chief disadvantage of constant-parameter models is that they cannot, in general, fit today's term structure of zero-coupon rates. In order to ensure that the short-rate dynamics are consistent with prices observed in the marketplace, we allow the parameters of the stochastic process to change through time. This section focuses on the application on *no-arbitrage pricing models*.[6] First, we assume that the changes in the short rate are normally distributed, and then, to prevent the possibility of negative interest rates, we assume the short rate is log-normally distributed (i.e., the logarithm of the short rate is normally distributed).

Normal Distribution

Suppose we consider the Vasicek model (20.4) with time-varying parameters, that is,

$$dr = a(t)[b(t) - r]dt + \sigma(t)dz \tag{20.7}$$

Note that the pull rate $a(t)$, the long-run mean $b(t)$, and the volatility of the short-rate $\sigma(t)$ are functions of time. The process (20.7) can again be approximated by a binomial process, that is,

$$r_{t+\Delta t} - r_t = \begin{cases} a(t)[b(t) - r]\Delta t + \sigma(t)\sqrt{\Delta t} & \text{with probability} = 1/2 \\ a(t)[b(t) - r]\Delta t - \sigma(t)\sqrt{\Delta t} & \text{with probability} = 1/2 \end{cases} \tag{20.8}$$

As before, we can see that the vertical distance between the two nodes emanating from $r_{i,j}$ in binomial lattice notation is $2\sigma(t)\sqrt{\Delta t}$, that is,

$$r_{i+1,j+1} - r_{i+1,j} = 2\sigma(i)\sqrt{\Delta t} \tag{20.9}$$

Note that the volatility parameter is the local volatility of the one-period short rate in one-period. Thus, if Δt is one year, $\sigma(0)$ is volatility of the one-year rate in one year, $\sigma(1)$ is volatility of the one-year rate in two years, $\sigma(2)$ is volatility of the one-year rate in three years, and so on.[7]

To make the binomial lattice procedure more tractable, we impose the restriction that the binomial lattice recombines (i.e., we set $r_{2,3} = r_{2,2}$ in Figure 20.1). This means

$$r_{2,3} = r_{1,2} + a(1)[b(1) - r_{1,2}]\Delta t - \sigma(1)\sqrt{\Delta t} \tag{20.10}$$

and

$$r_{2,2} = r_{1,1} + a(1)[b(1) - r_{1,1}]\Delta t + \sigma(1)\sqrt{\Delta t} \tag{20.11}$$

[6] The pioneering work on valuing interest rate derivatives using no-arbitrage pricing models is Ho and Lee (1986).

[7] In this section, we assume that the sequence of volatility estimates is known. In practice, they can be estimated from the prices of caps and floors.

Equating (20.10) and (20.11), rearranging, and simplifying, we get

$$r_{1,2} - r_{1,1} - a(1)(r_{1,2} - r_{1,1})\Delta t = 2\sigma(1)\sqrt{\Delta t} \tag{20.12}$$

Substituting (20.9) into (20.12), we get

$$2\sigma(0)\sqrt{\Delta t} - a(1)[2\sigma(0)\sqrt{\Delta t}]\Delta t = 2\sigma(1)\sqrt{\Delta t}$$

or

$$1 - a(1)\Delta t = \sigma(1)/\sigma(0) \tag{20.13}$$

Note that because we imposed the restriction that the binomial lattice recombines, the mean reversion parameter $a(1)$ is determined by the ratio of the ratio of the local volatility rates at adjacent time steps and need not be estimated separately.

We now turn to the computation of the binomial lattice in a no-arbitrage pricing framework. The key relation in computing the lattice efficiently is that we know the distance between adjacent vertical nodes at each time step (20.9). Begin by considering the possible levels of interest rates at the end of one period. As Figure 20.5 shows, there are two possibilities—$r_{1,1}$ and $r_{1,2} = r_{1,1} + 2\sigma(0)\sqrt{\Delta t}$ with equal probability. Since the volatility parameter and the time increment are known, identifying the numerical values of each of the two nodes is merely a matter of finding $r_{1,1}$. Suppose that the zero-coupon yield curve is described by the relation

$$r_i = 0.10 - 0.05e^{-0.18(T_i - 1)}$$

where t is measured in years and that the volatility rate is $\sigma(t) = 0.01$ for all t. Based on the zero-coupon yields, we compute the prices of one-year and two-

FIGURE 20.5 One-period binomial lattice for no-arbitrage pricing model assuming the short rate is normally distributed.

0	1
	$r_{1,2} = r_{1,1} + 2\sigma(0)\sqrt{\Delta t}$
$r_{0,1}$	
	$r_{1,1}$

where the nodes at time 1 are

$$r_{1,1} = r_{0,1} + a(0)[b(0) - r_{0,1}]\Delta t - \sigma(0)\sqrt{\Delta t} \text{ and}$$

$$\begin{aligned} r_{1,2} &= r_{0,1} + a(0)[b(0) - r_{0,1}]\Delta t - \sigma(0)\sqrt{\Delta t} \\ &= r_{1,1} + 2\sigma(0)\sqrt{\Delta t} \end{aligned}$$

year discount bonds. The one-year discount bond has a price of $DBP_1 = e^{-R(1)1} = e^{-0.05(1)} = 0.95123$ and the two-year discount bond has a price of $DBP_2 = e^{-R(2)2} = e^{-0.05824(2)} = 0.89005$, as summarized in this table:

Years to Maturity	Spot Rate	Discount Bond Price	Forward Discount Bond Price
1	5.000%	0.95123	
2	5.824%	0.89005	0.93569

Based on the prices the one-year and two-year discount factors, we can compute the forward price of a one-year discount bond in one year as $FBP_{1,1} = DBP_2/DBP_1 = 0.93569$. In the absence of costless arbitrage opportunities, it must be the case that the forward discount bond price from the zero-coupon yield curve must equal the expected discount value in the interest rate lattice. The value of $r_{1,1}$ can therefore be determined by solving

$$\begin{aligned} 0.93569 &= 0.5e^{-r_{1,1}\Delta t} + 0.5e^{(-r_{1,1} + 2\sigma(0)\sqrt{\Delta t})\Delta t} \\ &= 0.5e^{-r_{1,1}} + 0.5e^{-r_{1,1} + 0.02} \end{aligned}$$

The value can be determined iteratively using SOLVER. The value of $r_{1,1}$ is 5.6523%, and the value of $r_{1,2}$ is 7.6523%, as is shown in Figure 20.6.

We fill out the remaining lattice short-rate binomial lattice by using the same computational procedure recursively. Consider Figure 20.7, which shows the interest rate lattice over two periods. At the end of two periods, we have a

FIGURE 20.6 One-period binomial lattice for no-arbitrage pricing model assuming the short rate is normally distributed, the zero-coupon yield curve is $R(t) = 0.10 - 0.05e^{-0.18(t-1)}$ where t is measured in years, and the volatility rate is $\sigma(t) = 0.01$ for all t.

0	1
	7.6523%
5.0000%	
	5.6523%

FIGURE 20.7 Two-period binomial lattice for no-arbitrage pricing model assuming the short rate is normally distributed.

0	1	2
		$r_{2,3} = r_{2,1} + 4\sigma(1)\sqrt{\Delta t}$
	$r_{1,2} = r_{1,1} + 2\sigma(0)\sqrt{\Delta t}$	
$r_{0,1}$		$r_{2,2} = r_{2,1} + 2\sigma(1)\sqrt{\Delta t}$
	$r_{1,1}$	
		$r_{2,1}$

single unknown, $r_{2,1}$, because we know the distance between adjacent vertical nodes. To solve for its value, we must first compute the forward price of a one-year discount bond in two years, $FDB_{2,1}$. The zero-coupon yield curve tells us its value is 0.92415.

Years to Maturity	Spot Rate	Discount Bond Price	Forward Discount Bond Price
1	5.000%	0.95123	
2	5.824%	0.89005	0.93569
3	6.512%	0.82255	0.92415
4	7.086%	0.75318	0.91567

In the absence of costless arbitrage opportunities, the forward discount bond price from the zero-coupon yield curve must equal the expected discount price within the interest rate lattice. The value of $r_{2,1}$ can be determined by solving

$$0.92415 = 0.25e^{-r_{2,1}} + 0.5e^{-(r_{2,1} + 0.02)} + 0.25e^{-(r_{2,1} + 0.04)}$$

The solution to this equation is $r_{2,1} = 0.058976$. The rates at the middle and upper nodes are therefore 0.078976 and 0.098976, respectively. For year 4, the lowest minimum rate is identified by solving for

$$0.91567 = 0.125e^{-r_{3,1}} + 0.375e^{-(r_{3,1} + 0.02)} + 0.375e^{-(r_{3,1} + 0.04)} + 0.375e^{-(r_{3,1} + 0.06)}$$

The solution to this equation is $r_{3,1} = 0.058252$. The complete interest rate lattice over four periods is shown in Figure 20.8.

Note that, in the above computations, we need to identify the probabilities of arriving at binomial lattice node (i,j), where i is the number of the time step, and j is the number of the vertical node (with $j = 1$ being the lowest). The general formula for computing this probability is

$$p_{i,j} = \left(\frac{1}{2}\right)^i \frac{i!}{(j-1)!(i-j+1)!} \tag{20.14}$$

FIGURE 20.8 Three-period binomial lattice for no-arbitrage pricing model assuming the short rate is normally distributed, the zero-coupon yield curve is $R(t) = 0.10 - 0.05e^{-0.18(t-1)}$ where t is measured in years, and the volatility rate is $\sigma(t) = 0.01$ for all t.

0	1	2	3
			11.8252%
		9.8976%	
	7.6523%		9.8252%
5.0000%		7.8976%	
	5.6523%		7.8252%
		5.8976%	
			5.8252%

The probabilities of the nodes in the second time step are therefore

$$p_{2,1} = \left(\frac{1}{2}\right)^2 \frac{2!}{(1-1)!(2-1+1)!} = 0.25$$

$$p_{2,2} = \left(\frac{1}{2}\right)^2 \frac{2!}{(2-1)!(2-2+1)!} = 0.5$$

and

$$p_{2,3} = \left(\frac{1}{2}\right)^2 \frac{2!}{(3-1)!(2-3+1)!} = 0.25$$

The no-arbitrage pricing framework described above is interesting in a number of respects. First, the drift in the short rate is dictated by the zero-coupon yield curve. Note that at the end of period one, both possible short rates exceed the short rate a period earlier. This simply reflects the fact that the yield curve is strongly upward sloping. Second, the entire short-rate lattice can be summarized using two vectors. In the first, we record the lowest interest rate node for each time step, r_i,1, $i = 1, \ldots, n$. In the second, we record the local volatility rate, $\sigma(i)$, $i = 0, \ldots, n - 1$. Third, in computing the interest rate lattice, we required no specific knowledge of the pull rate $a(t)$ or the long-run mean reversion level $b(t)$. The long-run mean reversion is subsumed in matching of the forward discount bond price from the zero-coupon yield to the expected discount bond price procedure. The pull rate $a(t)$ is subsumed by the ratio of the local volatility rates in adjacent periods (see equation (20.14)).

ILLUSTRATION 20.1 Develop binomial lattice assuming short rate is normally distributed.

Assume the zero-coupon yield curve is

$$r_i = 0.12 - 0.06e^{-0.20(T_i - 1)}$$

and the local volatility function is $\sigma(i) = 0.015 - 0.00025\ln(1 + T_i)$. *Develop a four-period short-rate lattice where the short rate is a six-month rate.*

The first step in developing the interest rate lattice is to gather the problem information. Based on the zero-coupon yields, we can compute discount bond prices and forward discount bond prices. The problem information used as inputs in developing the interest rate lattice is as follows:

Years to Maturity	Spot Rate	Discount Bond Price	Forward Discount Bond Rate	Local Volatility Rate
0.5	5.369%	0.97351		1.099%
1	6.000%	0.94176	0.96739	1.027%
1.5	6.571%	0.90614	0.96217	0.971%
2	7.088%	0.86784	0.95773	0.925%
2.5	7.555%	0.82789	0.95397	

The next step is to identify the lowest interest rate node at each of the four time steps. At time 0, the lowest interest rate node is the spot rate 5.369%. At time 1, the lowest interest rate is determined by solving

$$0.96739 = 0.5e^{-r_{1,1}(0.5)} + 0.5e^{(-r_{1,1} + 2(0.01099)\sqrt{0.5})0.5}$$

The value of $r_{1,1}$ is 5.8557%. The vertical distance between adjacent nodes at the end of period one is $2(0.01099)\sqrt{0.5} = 0.015537$, so $r_{1,2}$ is 7.4094%.

The OPTVAL Library contains a function that determines the minimum short rate at each time step. The function has the syntax

OV_TS_LATTICE_RMIN(*fbp*,*v*,*tinc*,*nstep*,*nl*)

where *fbp* is the forward discount bond price, *v* is the local volatility rate, *tinc* is the length of each time step in years, *nstep* is the number of the current time step, and *nl* is an indicator variable instructing the function to assume the short rate is normally distributed ("N" or "n") or log-normally distributed ("L" or "l"). To perform the above computation, use

OV_TS_LATTICE_RMIN(0.96739,0.01099,.5,1,"n") = 0.058557

The minimum short rate at each time step is:

Time Step	Years to Maturity	Spot Rate	Discount Bond Price	Forward Discount Bond Price	Local Volatility Rate	Minimum Short Rate
0	0.5	5.369%	0.97351		1.099%	5.3690%
1	1	6.000%	0.94176	0.96739	1.027%	5.8557%
2	1.5	6.571%	0.90614	0.96217	0.971%	6.2636%
3	2	7.088%	0.86784	0.95773	0.925%	6.5814%
4	2.5	7.555%	0.82789	0.95397		6.8120%

The entire short-rate lattice over the two-year period is:

0	0.5	1	1.5	2
				12.0465%
			10.7007%	
		9.1676%		10.7379%
	7.4094%		9.3276%	
5.3690%		7.7156%		9.4293%
	5.8557%		7.9545%	
		6.2636%		8.1206%
			6.5814%	
				6.8120%

Log-Normal Distribution

The main problem with assuming interest rate changes are normally distributed is that there is some change that interest rates will become negative. A simple remedy to this problem is to assume that interest rates are log-normally distributed or put another way that the logarithm of the interest rate ln$r$ is normally distributed. The modifications to the no-arbitrage pricing procedure are straightforward. The binomial process is

$$\ln r_{t+\Delta t} - \ln r_t = \begin{cases} a(t)[b(t) - \ln r]\Delta t + \sigma(t)\sqrt{\Delta t} & \text{with probability} = 1/2 \\ a(t)[b(t) - \ln r]\Delta t - \sigma(t)\sqrt{\Delta t} & \text{with probability} = 1/2 \end{cases} \tag{20.15}$$

and the distance between adjacent vertical nodes in the binomial lattice is

$$\ln r_{i+1,j+1} - \ln r_{i+1,j} = 2\sigma(i)\sqrt{\Delta t} \tag{20.16}$$

Since we would prefer to have the lattice contain interest rates rather than the logarithm of interest rates, the log of interest rate spacing in (20.17) can be rewritten in interest rate form

$$\frac{r_{i+1,j+1}}{r_{i+1,j}} = e^{2\sigma(i)\sqrt{\Delta t}} \tag{20.17}$$

To illustrate the application of this binomial procedure, reconsider the rates zero-coupon yield curve of the running illustration. Furthermore, assume that the volatility rate is 20%.[8] The interest rates in year 2 are determined by solving

$$0.93569 = 0.5e^{-r_{2,1}(1)} + 0.5e^{-[r_{2,1} \times 2\sigma(1)]}$$

The solution for the minimum interest rate is 5.3421%. The volatility rate is 0.20, so the constant proportion between adjacent rates is 1.4918. The interest rate at the upper node at year 2 is therefore 0.053421 × 1.49182 = 0.079695. The full interest rate lattice under the log-normal assumption is provided in Figure 20.9.

FIGURE 20.9 Two-period binomial lattice for no-arbitrage pricing model assuming the short rate is log-normally distributed.

0	1	2
		$r_{2,3} = r_{2,1}e^{4\sigma(1)\sqrt{\Delta t}}$
	$r_{1,2} = r_{1,1}e^{2\sigma(0)\sqrt{\Delta t}}$	
$r_{0,1}$		$r_{2,2} = r_{2,1}e^{2\sigma(1)\sqrt{\Delta t}}$
	$r_{1,1}$	
		$r_{2,1}$

[8] Note that the volatility rate of the change in the logarithm of the interest rate is dramatically higher than the volatility rate of the change in interest rate.

BOND VALUATION

With the mechanics of generating an interest rate lattice in hand, we now turn to bond valuation. We start with the valuation of zero-coupon bonds, and then generalize the framework to handle coupon-bearing bonds. We then show how the framework can be modified to handle bonds with embedded options such as callable bonds and putable bonds.

Zero-Coupon Bonds

To value options on bonds in a framework with short-term interest risk as the underlying source of uncertainty requires that we first create a bond price lattice. In order to do so, we extend the interest rate lattice to the end of the bond's life (which may be well beyond the option's life). To illustrate, consider a 4-year discount bond. In year 4, the bond matures with a payment of principal. Assume the principal is 100. In year 3, the short-term interest in the uppermost node is 0.152051. The value of the bond at that interest rate is $100e^{-0.152051(1)} = 85.894$. At the second uppermost node, the bond's value is $100e^{-0.101923(1)} = 90.310$, and so on.

To compute the bond's value in year 2, we must include the probabilities of upward and downward interest rate movements. The value of the bond at the uppermost node in year 2 is computed as

$$e^{-0.113447(1)}\left[\frac{1}{2}\times 85.894+\frac{1}{2}\times 90.310\right] = 78.653$$

The value of the bond at the second uppermost node is

$$e^{-0.076046(1)}\left[\frac{1}{2}\times 90.310+\frac{1}{2}\times 93.396\right] = 84.127$$

The price lattice of the four-year discount bond is shown in Figure 20.10.

FIGURE 20.10 Three-period binomial lattice for no-arbitrage pricing model assuming the short rate is log-normally distributed, the zero-coupon yield curve is $R(t) = 0.10 - 0.05e^{-0.18(t-1)}$ where t is measured in years, and the volatility rate is $\sigma(t) = 0.20$ for all t.

0	1	2	3
			15.2051%
		11.3447%	
	7.9695%		10.1923%
5.0000%		7.6046%	
	5.3421%		6.8321%
		5.0975%	
			4.5797%

FIGURE 20.11 Valuation of a four-year zero-coupon bond using a no-arbitrage pricing model that assumes the short rate is log-normally distributed, the zero-coupon yield curve is $R(t) = 0.10 - 0.05e^{-0.18(t-1)}$ where t is measured in years, and the volatility rate is $\sigma(t) = 0.20$ for all t.

0	1	2	3	4
				100
			85.894	
		78.653		100
	75.617		90.310	
75.392		85.127		100
	82.897		93.396	
		89.765		100
			95.524	
				100

Coupon-Bearing Bonds

The interest rate lattice can be used to value all sorts of bonds. To illustrate its generality, assume that we want to value a four-year coupon-bearing bond with annual coupon payments equal to 6. Again we start at the end of the bond's life. In year 4, the bond matures with a coupon payment of 6 and a repayment of principal of 100. In year 3, the short-term interest in the uppermost node is 0.152051. The value of the bond at that interest rate is $106e^{-0.152051(1)} = 97.048$. At the second uppermost node, the bond's value is $106e^{-0.101923(1)} = 101.729$, and so on.

As we proceed backward in time, we must add in the coupon payments. The value of the bond at the uppermost node in year 2 is

$$e^{-0.113447(1)}\left[\frac{1}{2}\times 97.048+\frac{1}{2}\times 101.729\right]+6 = 94.729$$

and at the second uppermost node is

$$e^{-0.076046(1)}\left[\frac{1}{2}\times 101.729+\frac{1}{2}\times 105.000\right]+6 = 101.795$$

The price lattice of the four-year coupon-bearing bond is provided in Figure 20.12.

Callable Bonds

A callable bond is a coupon-bearing bond that allows its issuer to retire the bond before its stated maturity. In general, the call dates of the bond are coupon-payment dates, and the amount that bondholders will be paid is the par value of the bond plus the current coupon.

FIGURE 20.12 Valuation of a four-year 6% coupon-bearing bond using a no-arbitrage pricing model that assumes the short rate is log-normally distributed, the zero-coupon yield curve is $R(t) = 0.10 - 0.05e^{-0.18(t-1)}$ where t is measured in years, and the volatility rate is $\sigma(t) = 0.20$ for all t.

0	1	2	3	4
				106
			97.048	
		94.729		106
	96.735		101.729	
95.899		101.795		106
	104.897		105.000	
		106.853		106
			107.255	
				106

Consider the 6% coupon-bearing bond valued in Figure 20.12. To value the bond, we began at the end of the bond's life and worked backwards, taking the present value of the expected future value of the bond one node at a time. In looking at the values reported at time step 3, note that at the bottom node, the value of the bond is 107.255. If this bond was callable, the issuer would call the bond at this node because calling it would cost 106 while waiting one more period would cost 107.255. Thus in valuing this callable bond, we replace the value at this node with 106, as shown in Figure 20.13, Panel A. Note that the value of the bond at the lowest node at time step 2 has changed from 106.853 in Figure 20.12 to 106.257 in Figure 20.13, Panel A, reflecting the call feature of the bond. But if interest rates evolved in a manner that the firm would find itself at the lowest node at time step 2, it would call the bond since the present value of its expected future value exceeds its immediate redemption value, 106. Again we replace the computed value of the bond, as shown in Figure 20.13, Panel B. Working backward to time 0, we find that the value of the callable bond is 95.707. Comparing this bond value to the noncallable coupon-bearing bond value, we find that from the firm's perspective, the value of the call feature is 0.202.

Putable Bonds

A putable bond permits the bondholder to sell the bond back to the issuer, usually at the par value of the bond. This put gives the bondholder some protection from loss of principal due to higher interest rates or credit deterioration of the issuer. Putable bonds can be valued straightforwardly using our interest rate lattice procedure. Suppose, for example, that the coupon-bearing bond shown in Figure 20.12 is putable at par by the bondholder. Since the put will be exercised only when the value of the bond falls below par value, we replace only the uppermost node at time step 3, as shown in Figure 20.14, Panel A. Moving back one time step, we see also that the bond will be put back to the issuer at the uppermost node. Therefore as shown in Figure 20.14, Panel B, we replace the

FIGURE 20.13 Valuation of a four-year 6% coupon-bearing callable bond using a no-arbitrage pricing model that assumes the short rate is log-normally distributed, the zero-coupon yield curve is $R(t) = 0.10 - 0.05e^{-0.18(t-1)}$ where t is measured in years, and the volatility rate is $\sigma(t) = 0.20$ for all t.

Panel A:

0	1	2	3	4
				106
			97.048	
		94.729		106
	96.735		101.729	
95.765		101.795		106
	104.615		105.000	
		106.257		106
			106.000	
				106

Panel B:

0	1	2	3	4
				106
			97.048	
		94.729		106
	96.735		101.729	
95.707		101.795		106
	104.493		105.000	
		106.000		106
			106.000	
				106

uppermost node with a value of 100. Finally, at the end of time step 1, the bondholder will exercise his option at the uppermost node, so, again, we replace the computed value of the bond with the exercise proceeds of 100. The value of the putable bond is 97.452. The value of the nonputable coupon-bearing bond is 95.899. The value of the embedded put is therefore 1.447.

BOND OPTION VALUATION

The interest rate lattice procedure can also be used to value bond options. Assume, for example, that we want to value a two-year European-style put option with an exercise price of 100. Also assume that the option expires just after the coupon is paid in year 2. In year 2, therefore, the put's value will be depend on the ex-coupon bond price, which is the price reported in year 2 less 6. Given that the

FIGURE 20.14 Valuation of a four-year 6% coupon-bearing puttable bond using a no-arbitrage pricing model that assumes the short rate is log-normally distributed, the zero-coupon yield curve is $R(t) = 0.10 - 0.05e^{-0.18(t-1)}$ where t is measured in years, and the volatility rate is $\sigma(t) = 0.20$ for all t.

Panel A:

0	1	2	3	4
				106
			100.000	
		96.047		106
	97.343		101.729	
96.189		101.795		106
	104.897		105.000	
		106.853		106
			107.255	
				106

Panel B:

0	1	2	3	4
				106
			100.000	
		100.000		106
	99.169		101.729	
95.707		101.795		106
	104.897		105.000	
		106.853		106
			107.255	
				106

Panel C:

0	1	2	3	4
				106
			100.000	
		100.000		106
	100.000		101.729	
97.452		101.795		106
	104.897		105.000	
		106.853		106
			107.255	
				106

put is expiring, its values are given by the lower boundary condition max(0,X – B), where X is the exercise price of the option and B is the bond price.

Time	0	1	2
		max(0,100 – 88.734) =	**11.266** 88.734
		max(0,100 – 94.798) =	**4.202** 95.798
		max(0,100 – 100.855) =	**0.000** 100.855

The value of the option in year 1 is the present value of the expected future value. At the uppermost node, the computation is

$$e^{-0.0797(1)}\left[\frac{1}{2}\times 11.266+\frac{1}{2}\times 4.202\right] = 7.142$$

At the lowermost node, the computation is

$$e^{-0.0534(1)}\left[\frac{1}{2}\times 4.202+\frac{1}{2}\times 0.000\right] = 1.991$$

The value of the put today is 4.567, as is shown in this figure:

Time	0	1	2
			11.266 **88.734**
		7.142	
	4.567		**4.202** **95.798**
		1.991	
			0.000 **100.855**

SUMMARY

The purpose of this chapter is modest—to develop a binomial procedure for valuing interest rate derivative contracts where the short-term interest rate ("short rate") is the single underlying source of interest rate uncertainty. To begin, we discuss a number of constant-parameter, short-rate processes to lay a foundation for interest rate behavior. While these models are often useful in developing economic intuition regarding interest rate behavior, they produce zero-coupon bond values that are different from the observed market prices, seemingly giving rise to arbitrage opportunities. Consequently, we next turn to no-arbitrage pricing models. These models adjust the parameters of the interest rate process in a manner that produces bond (and interest rate derivatives contract) values equal to observed prices. With the mechanics of no-arbitrage pricing in hand, we then turn to valuing zero-coupon and coupon-bearing bonds, callable bonds, putable bonds, and bond options. Be forewarned, however. While the valuation framework provided in this chapter is intuitive and commonly applied in practice, it only begins to scratch the surface of the literature focused on no-arbitrage interest rate models. This literature is deep in multifactor theoretical models of interest rate movements and numerical procedures for calibrating the interest rate models and valuing interest rate derivatives.

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