Building on Black-Scholes

The economic insights of Black, Scholes and Merton laid the foundations for a quarter-century of theoretical work. Robert Whaley presents a brief history of option pricing

"Because options are specialised and relatively unimportant financial securities, the amount of time and space devoted to the development of a pricing theory might be questioned." (Robert Merton, Bell Journal of Economics and Management Science, spring 1973)

Twenty-five years ago, no-one – not even, apparently, one of our recent Nobel Laureates – could have imagined the changes that were about to occur in the direction of finance theory and the development of the financial products industry. The seeds of change were contained in option valuation research being conducted by Fischer Black, Robert Merton and Myron Scholes. The purpose of this article is to describe how their pioneering work provided the foundation for modern-day option valuation theory and the structure for financial products whose notional value now reportedly exceeds $70 trillion worldwide.

Background

Valuing claims to income streams is one of the central problems of finance. The exercise is straightforward conceptually – it amounts to identifying the amount and the timing of the expected cashflows from holding the claim and then discounting them back to the present. Valuing a European-style call option, therefore, requires that we estimate (a) the mean of the call option's payout distribution on the day it expires and (b) the discount rate to apply to the option's expected terminal payout.

The first known attempt to value a call option occurred near the turn of the century. In his dissertation, "Theory of Speculation", Bachelier (1900) values a call option by assuming that the underlying asset price follows arithmetic Brownian motion. While applying Brownian motion in any context was remarkable for its day[1], applying it to describe asset price movements has the unfortunate implication that the asset price may become negative.

To circumvent this problem, Sprenkle (1961) and Samuelson (1965) attempted to value options under the assumption that asset prices follow *geometric* Brownian motion. By letting asset prices have multiplicative, rather than additive, fluctuations through time, the asset price distribution at the option's expiry is lognormal, rather than normal, and the prospect of the asset price becoming negative is eliminated. Under lognormality, Sprenkle and Samuelson showed that the call option valuation formula has the form:

$$c = e^{-\alpha_c T}\left[Se^{\alpha_s T}N(d_1) - XN(d_2)\right] \tag{1}$$

where:

$$d_1 = \frac{\ln(S/X) + \left(\alpha_S + 0.5\sigma^2\right)T}{\sigma\sqrt{T}},\quad d_2 = d_1 - \sigma\sqrt{T}$$

and α_s and α_c are the expected risk-adjusted rates of price appreciation for the asset and the call respectively, σ is the asset's volatility rate, S is the current asset price, X is the option's exercise price and T is the option's time to expiry. The expression $N(.)$ is the cumulative univariate normal probability function.

The structure of (1) shows that the call option's value is the present value of its expected terminal value. The expected terminal value depends on various factors, including the expected growth rate of the asset price, α_s. The call is a claim to buy the asset, and the expected asset price at the option's expiry is $Se^{\alpha_s T}$. The expression $Se^{\alpha_s T}N(d_1)$ is the expected asset price, conditional on the asset price exceeding the exercise price at the option's expiry. The expression $XN(d_2)$ is the expected exercise cost, ie, the exercise price multiplied by the probability that the option will be exercised.

As elegant and precise as formula (1) appears, it is not very useful. To implement the formula requires estimates of the risk-adjusted rates of price appreciation for both the asset and the option. To estimate these values

[1] *Samuelson (1965, page 13) notes that Bachelier "...discovered the mathematical theory of Brownian motion five years before Einstein's classic paper"*

precisely is problematic. Moreover, as if the estimation problems are not vexing enough, one must still consider the fact that the return of the call depends on the asset's return as well as the passage of time.

The Black-Scholes-Merton theory

The Sprenkle/Samuelson contributions set the stage. The breakthrough came in the early 1970s. Black & Scholes (1973) and Merton (1973) showed that, as long as a risk-free hedge may be formed between the option and its underlying asset, the value of an option relative to the asset will be the same for all investors, regardless of their risk preferences.

The intuition underlying the risk-free hedge argument is simple. Consider an at-the-money European-style call option that allows its holder to buy one unit of an asset in one month at an exercise price of $40. For the sake of simplicity, suppose that, at the end of one month, the asset price will be either $45 or $35. Now, consider selling call options against the unit investment in the asset. At the end of the month, each call will have a value of $5 or $0, depending on whether the asset price is $45 or $35. Under this scenario, selling two call options against each unit of the asset will create a terminal portfolio value of $35, regardless of the level of asset price. Since the terminal portfolio value is certain, the value of the portfolio today must be $35 discounted at the risk-free rate of interest. If the risk-free rate of interest is 1%, the current value of the portfolio must be $34.65, and the current value of the call $2.675, ie, ($40–34.65)/2. If the observed price of the call is above (below) its theoretical level of $2.675, risk-free arbitrage profits are possible by selling the call and buying (selling) a portfolio consisting of a long position in a half unit of the asset and a short position of $17.325 in risk-free bonds. In equilibrium, no such arbitrage opportunities can exist.

The Black-Scholes-Merton (BSM) model is the continuous-time analogue of this illustration. If the asset price follows geometric Brownian motion, a risk-free hedge can be formed between the option and its underlying asset, implying that the payout of a European-style call can be identically duplicated by a portfolio consisting of the asset and risk-free bonds.[2] Put simply, option value does not depend on the asset's expected return and is therefore independent of investor risk preferences. The value of an option is the same for a risk-neutral investor as it is for a risk-averse investor. Without loss of generality, therefore, options can be valued in a risk-neutral world where expected asset returns and expected option returns all equal the risk-free rate of interest.

Analytical formulas

The BSM option valuation theory goes well beyond the "formula" bearing their names. Their key economic insight is that if a risk-free hedge between the option and its underlying asset may be formed, risk-neutral valuation may be applied. This applies to any option. Sometimes the option's payout contingencies are sufficiently straightforward that an analytical formula for the option's value can be found. This is the case for the "BSM formula", which provides the value of a standard European-style call option. Options with analytical formulas are the focus of this section. Sometimes the option's payout contingencies are so complex that analytical solutions are not possible. In these cases, the BSM theory continues to apply, although option values must be calculated numerically. Numerical methods, as applied to option valuation problems, are the focus of the next section.

The BSM formula for a European-style call follows directly from the work of Sprenkle and Samuelson. In a risk-neutral world, all assets (including options) have an expected rate of return equal to the risk-free interest rate, r. That is not to say, however, that all assets have the same expected rate of price appreciation. Some assets pay out income in the form of dividends or coupon interest. If the asset's income is modelled as a constant, continuous proportion of the asset price, the expected rate of price appreciation on the asset, denoted b, equals the interest rate less the cash disbursement rate.

Retaining the assumption of risk-neutrality, we now return to formula (1) and substitute appropriate price appreciation rates. The expected rate of price appreciation of the call, α_c, is set equal to the risk-free rate, r, since options are not income-producing assets. The expected rate of price appreciation of the asset, α_s, is set equal to b. The BSM formula for the value of a European-style call option is:

$$c = e^{-rT}\left[Se^{bT}N(d_1) - XN(d_2)\right] \tag{2}$$

Note that with (2), there is no need either to estimate the risk premiums of the call and the asset or to model how the call's risk premium changes through time.

The BSM formula covers a wide range of underlying assets. The distinction between the valuation problems described below rests in the asset's risk-neutral price appreciation parameter, b.

☐ *Non-dividend-paying stock options.* The most well-known option valuation problem is that of valuing options on non-dividend-paying stocks. This is, in fact, the valuation problem addressed by Black & Scholes (1973). With no dividends paid on the underlying stock, the expected price appreciation rate of the stock equals the risk-free rate of interest, and the call option valuation equation becomes the familiar "Black-Scholes formula":

$$c = SN(d_1) - Xe^{-rT}N(d_2)$$

[2] This "law of one price" argument is no stranger to financial economics. It is the fundamental theoretical underpinning of the Nobel prize-winning corporate finance theory of Modigliani & Miller (1958) and Miller & Modigliani (1961)

☐ *Constant-dividend-yield stock options.* Merton (1973) generalises stock option valuation by assuming that stocks pay dividends at a constant, continuous dividend yield. The "Merton model" is simply equation (2), replacing b with the difference between the risk-free rate and the stock's dividend yield rate.

☐ *Futures options.* Black (1976) values options on futures. In a risk-neutral world with constant interest rates, the expected rate of price appreciation on a futures contract, because it involves no cash outlay, is zero. Substituting into (2) provides what is commonly known in the futures industry as the "Black model".

☐ *Futures-style futures options.* Following the work of Black, Asay (1982) values futures-style futures options. Such options trade on various exchanges, including the London International Financial Futures and Options Exchange, and have the distinguishing feature that the option premium is not paid upfront. Instead, the option position is marked to market in the same manner as the underlying futures. To value this option, we not only set $b = 0$ inside the square brackets to reflect the zero expected rate of price appreciation on the futures, but also set $r = 0$ outside the square brackets because an option requiring zero investment upfront must have an expected price appreciation equal to zero. The resulting formula is called the "Asay model".

☐ *Foreign currency options.* Finally, Garman & Kohlhagen (1983) value options on foreign currency. Here, the expected rate of price appreciation of a foreign currency equals the domestic rate of interest less the foreign rate of interest. Substituting equation (2) becomes the "Garman-Kohlhagen model". Earlier, we identified the key contribution of the BSM model as being the recognition that a risk-free hedge can be formed between the option and the underlying asset. Consequently, the payouts of a call option can be replicated with a portfolio consisting of the asset and some risk-free bonds.

The BSM formula provides the composition of the asset/bond portfolio that mimics the payouts of the call. To replicate a long call position, we buy $e^{(b-r)T}N(d_1)$ units of the asset, each unit with price S, and sell $N(d_2)$ units of risk-free bonds, each unit with price Xe^{-rT}. As time passes and the asset price moves, the units invested in the asset and risk-free bonds will change. But, with continuous rebalancing, the portfolio's payouts will be identical to those of the call.

☐ *Dynamic portfolio insurance*[3]. Dynamic replication is at the heart of one of the most popular financial products of the 1980s – dynamic portfolio insurance. Because long-term index put options were not traded at the time, stock portfolio managers had to create their own insurance by dynamically rebalancing a portfolio consisting of stocks and risk-free bonds. The mechanism for identifying the portfolio weights is given by the BSM put option formula:

$$p = Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1)$$

The objective is to create an "insured" portfolio whose payouts mimic the portfolio $Se^{(b-r)T} + p$. Substituting the BSM put formula, we find:

[3] *For a lucid description of portfolio insurance, see Rubinstein (1985)*

$$Se^{(b-r)T} + p = Se^{(b-r)T} + Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1)$$
$$= Se^{(b-r)T}N(d_1) + Xe^{-rT}N(-d_2)$$

Hence, a dynamically insured portfolio has $e^{(b-r)T}N(d_1)$ units of stocks and $N(-d_2)$ units of risk-free bonds. The weights show that as stock prices rise, funds are transferred from bonds to stocks and vice versa.

□ *The first exotic option?* The valuation-by-replication technique can also be applied in a static context. Indeed, many multiple contingency financial products such as caps, collars and floors are valued as portfolios of standard options. Before considering such "exotics", however, it is worthwhile to note that a standard call option is, itself, exotic. Consider a portfolio that consists of (a) a long position in an asset-or-nothing call that pays the asset price at expiry if the asset price exceeds X and (b) a short position in a cash-or-nothing call that pays X if the asset price exceeds X.[4] Under the assumptions of risk-neutrality and lognormally distributed asset prices, the value of the asset-or-nothing call is $Se^{(b-r)T}N(d_1)$, and the value of the cash-or-nothing call option is $Xe^{-rT}N(d_2)$. Combining these option values, we get the BSM formula (2).

The BSM option valuation framework has been extended in several important ways. Some involve more complex claims on a single underlying asset. Here, we focus on the valuation of such claims. Others, which we examine below, involve claims on two or more underlying assets.

□ *Compound options.* An important extension of the BSM model that falls in the single underlying asset category is the compound option valuation theory developed by Geske (1979a). Compound options are options on options. A call on a call, for example, provides its holder with the right to buy a call on the underlying asset at some future date. Geske shows that, if these options are European-style, valuation formulas can be derived.

□ *American-style call options on dividend-paying stocks.* The Geske (1979a) compound option model has been applied in other contexts. Roll (1977), Geske (1979b) and Whaley (1981), for example, develop a formula for valuing an American-style call option on a stock with known discrete dividends. If a stock pays a cash dividend during the call's life, it may be optimal to exercise the call early, just prior to dividend payment. An American-style call on a dividend-paying stock, therefore, can be modelled as a compound option providing its holder with the right, on the ex-dividend date, either to exercise early and collect the dividend, or to leave the position open.

□ *Chooser options.* Rubinstein (1991) uses the compound option framework to value the "chooser" or "as-you-like-it" options traded in the over-the-counter market. The holder of a chooser option has the right to decide at some future date whether the option is a call or a put. The call and the put usually have the same exercise price and the same time remaining to expiry.

□ *Bear market warrants with a periodic reset.* Gray & Whaley (1997) use the compound option framework to value yet another type of contingent claim, S&P 500 bear market warrants with a periodic reset traded at the Chicago Board Options Exchange and the New York Stock Exchange. The warrants are originally issued as at-the-money put options but have the distinguishing feature that if the underlying index level is above the original exercise on some pre-specified future date, the exercise price of the warrant is reset at the then-prevailing index level. These warrants offer an intriguing form of portfolio insurance whose floor value adjusts automatically as the index level rises. The structure of the valuation problem is again a compound option, and Gray & Whaley provide the valuation formula.

□ *Lookback options.* A lookback option is another exotic that has only one underlying source of price uncertainty. Such an option's exercise price is determined at the end of the its life. For a call, the exercise price is set equal to the lowest price that the asset reached during the life of the option; for a put, the exercise price equals the highest asset price. These "buy at the low" and "sell at the high" options can be valued analytically. Formulas are provided in Goldman, Sosin & Gatto (1979).

□ *Barrier options.* Barrier options are the final type of option in this category that I will discuss. Barrier options are options that either cease to exist or come into existence when some pre-defined asset price barrier is hit during the option's life. A down-and-out call, for example, is a call that gets "knocked out" when the asset price falls to some pre-specified level prior to the option's expiry. Rubinstein & Reiner (1991) provide valuation equations for a large family of barrier options.

The BSM option valuation framework has also been extended to include multiple underlying assets. As long as each asset is traded, the BSM risk-free hedge argument remains intact and risk-neutral valuation is permitted without loss of generality.

□ *Exchange options.* The first important development along this line was by Margrabe (1978). He derives a valuation formula for an exchange option, ie, the right to exchange one risky asset or asset for another. The BSM formula is a special case of the Margrabe formula in the sense that if the call is in-the-money at expiry the option holder exchanges risk-free bonds for the asset.

□ *Options on the minimum and the maximum.* Stulz (1982) and Johnson (1987) derive valuation formulas for options on the maximum and the minimum of two or more risky assets. Many of the exchange-traded futures contracts can be valued as an option on the minimum. The Chicago Board of Trade's Treasury bond futures, for example, provide the seller with the right to deliver the cheapest of a number of deliverable T-bond issues.

Approximation methods

Many option valuation problems do not have explicit closed-form solutions. Probably the best known example is the valuation of American-style options. With American-style options, the option holder has an infinite number of exercise opportunities between the current date and the option's expiry date, making the problem intractable from a mathematical standpoint.[5] But many other examples exist. Hundreds of different types of exotic options trade in the OTC market, and many, if not most, do not have analytical formulas. Nonetheless, they can all be valued accurately using the BSM model. If a risk-free hedge can be formed between the option and the underlying asset, the BSM risk-neutral valuation theory can be applied, albeit through the use of numerical methods. Below, I describe three types of commonly-applied approximation methods.

A number of numerical methods for valuing options are lattice-based. These methods replace the BSM assumption that asset price moves smoothly and continuously through time with an assumption that the asset price moves in discrete jumps over discrete intervals during the option's life.

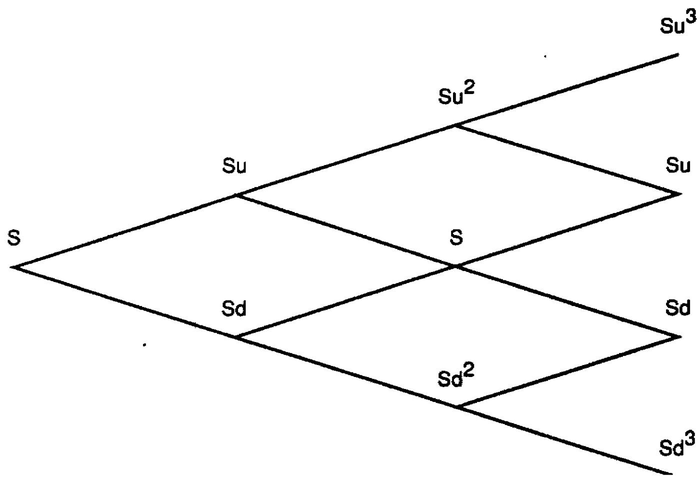
□ *Binomial method.* Perhaps the best-known lattice-based method is the binomial method, developed independently by Cox, Ross & Rubinstein (1979) and Rendleman & Bartter (1979). In the binomial method, the asset price jumps up or down, by a fixed proportion, at each of a number of discrete time steps during the option's life. The length of each time step, Δt, is determined when the user specifies the number of time steps, n, ie, $\Delta t = T/n$. The greater the number of time steps, the more precise the method. The cost of the increased precision, however, is computational speed. With n time steps, 2^n asset price paths over the life of the option are considered. With 20 time steps, this means more than 1 million paths.

In describing the steps of the binomial method, I will use, as a running illustration, the valuation of an American-style option. The first step of the binomial method is to enumerate the possible paths that the asset price may take between now and the option's expiry. The up-step and down-step coefficients, u and d, and their respective probabilities of an up-step

[4] *Asset-or-nothing and cash-or-nothing options are commonly referred to as "binary" or "digital" options, and, themselves, are generally considered to be "exotics"*

[5] *An exception is, of course, an American-style call option on an asset whose risk-neutral rate of price appreciation is greater than or equal to the risk-free rate of interest*

Binomial tree



and a down-step, p_u and p_d, are set in such a manner that (a) the mean and variance of the discrete distribution are equal to the mean and the variance of the lognormal distribution, (b) the asset price tree "recombines", and (c) the probabilities of an up-step and a down-step add up to one. One set of coefficients and probabilities that satisfies these conditions is:

$$u = e^{\sigma\sqrt{\Delta t}}, d = 1 / u, p_u = \left(e^{b\Delta t} - d\right) / (u - d), p_d = 1 - p_u$$

The enumeration procedure begins with the current asset price and moves forward through time. With n time steps, there will be $n+1$ terminal asset price nodes, as shown in the figure.

The procedure then moves to the end of the option's life and begins to work backwards. At the end of the option's life, the option value at each asset price node is simply the option's intrinsic value. Once the option values at all nodes at time n are identified, the procedure steps backward one time step.

At time $n-1$, the value of the option at each node is calculated by taking the present value of the expected future value of the option. The expected future value is simply the probability of an up-step times the option's value if the asset price steps up plus the probability of a down-step times the option's value if the asset price steps down. The discount rate in the present value computation is the risk-free rate of interest, as we continue to operate within the Black-Scholes-Merton framework.

Before proceeding back another time step, it is necessary to check if any of the calculated option values at time $n-1$ are below their early-exercise proceeds. Since we know the asset price underlying each calculated option price, it is a small matter to compare the computed option value with the option's early exercise proceeds at each node. If the early exercise proceeds exceed the calculated value, we replace the calculated value with the exercise proceeds. The interpretation is, of course, that if the option holder finds himself standing at that time in the option's life with the underlying asset priced at that level, he will exercise his option. If proceeds are less, the calculated value is left undisturbed. Note that, if the check of the early exercise condition is not performed, the binomial method will produce an approximate value for a European-style option.[6]

The procedure now takes another step back in time, repeats the calculations of all nodes, and then checks for early exercise. The procedure is repeated again and again until only a single node remains at time 0. This node will contain the value of the American-style option, as approximated by the binomial method.

The binomial method has wide applicability. Aside from the American-style option feature, which is easily incorporated within the framework, the binomial method can be used to value many types of exotic options. Knock-out options, for example, can be valued using this technique. We simply impose a different check on the calculated option values at the nodes of the intermediate time steps between 0 and n, ie, if the underlying asset price falls below the option's barrier, the option value at that node is set equal to zero. The method can also be extended to handle multiple sources of asset price uncertainty. Boyle, Evnine & Gibbs (1989) adapt the binomial procedure to handle exotics with multiple sources of uncertainty, including options on the minimum and maximum, spread options and so on.

☐ *Trinomial method.* The trinomial method is another popular lattice-based method. As outlined by Boyle (1988), this allows the asset to move up, down or stay the same at each time increment. Again, the parameters of the discrete distribution are chosen in a manner consistent with the lognormal distribution, and the procedure begins at the end of the option's life and works backward. By having three branches instead of two, the trinomial method provides greater accuracy than the binomial method for a given number of time steps. The cost is, of course, that the greater the number of branches, the slower the computational speed.

☐ *Finite difference method.* The explicit finite difference method was the first lattice-based procedure to be applied to option valuation. Schwartz (1977) applied it to value warrants and Brennan & Schwartz (1977) applied it to value American-style put options on common stocks. The finite difference method is similar to the trinomial method in the sense that the asset price moves up, down or stays the same at each time step during the option's life. The difference in the techniques arises only from how the price increments and the probabilities are set. In addition, finite difference methods calculate an entire rectangle of node values rather than simply a tree.

Boyle (1977) introduces Monte Carlo simulation to value options. Like

6 Indeed, a useful way to gauge the approximation error of the various numerical methods is to implement them on valuation problems for which there is an analytical formula the lattice-based procedures, the technique involves simulating possible paths that the asset price may take over the life of the option. And, again, the simulation is performed in a manner consistent with the lognormal asset price process. To value a European-style option, each sample run is used to produce a terminal asset price, which, in turn, is used to determine the terminal option value. With repeated sample runs, a distribution of terminal options values is obtained, and the expected terminal option value may be calculated. This expected value is then discounted to the present to value the option. An advantage of the Monte Carlo method is that the degree of valuation error can be assessed directly using the standard error of the estimate. The standard error equals the standard deviation of the terminal option values divided by the square root of the number of trials.

Another advantage of the Monte Carlo technique is its flexibility. Since the path of the asset price beginning at time 0 and continuing through the life of the option is observed, the technique is well-suited for handling barrier-style options, Asian-style options, Bermuda-style options and the like. Moreover, it can easily be adapted to handle multiple sources of price uncertainty. The technique's chief disadvantage is that it can be applied only when the option payout does not depend on its value at future points in time. This eliminates the possibility of applying the technique to American-style option valuation, where the decision to exercise early depends on the value of the option that will be forfeit.

□ *Compound option approximation*. The quasi-analytical methods for option valuation are quite different from the procedures that attempt to describe asset price paths. Geske & Johnson (1984), for example, use a Geske (1979a) compound option model to develop an approximate value for an American-style option. The approach is intuitively appealing. An American-style option, after all, is a compound option with an infinite number of early exercise opportunities. While valuing an option in this way makes intuitive sense, the problem is intractable from a computational standpoint. The Geske-Johnson insight is that, although we cannot value an option

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with an infinite number of early exercise opportunities, we can extrapolate its value by valuing a sequence of "pseudo-American" options with zero, one, two and perhaps more early exercise opportunities at discrete, equally spaced intervals during the option's life. The advantage that this offers is that each of these options can be valued analytically. With each new option added to the sequence, however, the valuation of a higher-order multivariate normal integral is required. With no early exercise opportunities, only a univariate function is required, however; with one early exercise opportunity, a bivariate function is required, with two opportunities, a trivariate, and so on. The more of these options used in the series, the greater the precision in approximating the limiting value of the sequence. The cost of increased precision is that higher-order multivariate integral valuations are time-consuming computationally.

☐ *Quadratic approximation.* Barone-Adesi & Whaley (1987) present a quadratic approximation. Their approach, based on the work of MacMillan (1986), separates the value of an American-style option into two components: the European-style option value and an early exercise premium. Since the BSM formula provides the value of the European-style option, they focus on approximating the value of the early exercise premium. By imposing a subtle change to the BSM partial differential equation, they obtain an analytical expression for the early exercise premium, which they then add to the European-style option value, thereby providing an approximation of the American-style option value. The advantage of the quadratic approximation method are speed and accuracy.

For many years, the search for quasi-analytical approximations was an important research pursuit. Using lattice-based procedures or Monte Carlo simulation was impractical in real-time applications. This pursuit has become much less critical, thanks to Moore's Law. In April 1965, Gordon Moore, an engineer and co-founder of Intel, predicted that integrated circuit complexity would double every two years. The prediction has been surprisingly accurate. In the late 1970s, when the lattice-based and Monte Carlo simulation methods were first applied to option valuation problems, Intel's most advanced microprocessor technology was the 8086 chip. Today, the Pentium Pro microprocessor is more than 1,000 times faster, and the impracticality of lattice-based and simulation-based methods has been substantially reduced.

Generalisations

The generalisations of the BSM option valuation theory focus on the assumed asset price dynamics. Some examine the valuation implications of modelling the local volatility rate as a deterministic function of the asset price or time or both. Others examine the valuation implications when volatility, like asset price, is stochastic.

Under the assumption that the local volatility rate is a deterministic function of time or the asset price or both, the BSM risk-free hedge mechanics are preserved so risk-neutral valuation remains possible. The simplest in this class of models is the case where the local volatility rate is a deterministic function of time. For this case, Merton (1973) shows that the valuation equation for a European-style call option is the BSM formula (2), where the volatility parameter is the average local volatility rate over the life of the option.

Other models focus on the relation between asset price and volatility and attempt to account for the empirical fact that, in at least some markets, volatility varies inversely with the level of asset price. One such model is the constant elasticity of variance model proposed by Cox & Ross (1976). In this model, asset price volatility is σS^{a-1}, where a falls in the range $0 \leq a \leq 1$. Where $a = 1$, volatility is constant, and a European-style call option can be valued analytically using the BSM formula (2). Where $a = 0$, volatility is inversely proportional to the asset price, and a European-style call option can also be valued analytically using a formula called the "absolute diffusion model".[7] For the general case where $0 < a < 1$, analytical solutions are not possible; however, valuation can be handled straightforwardly using lattice-based or Monte Carlo simulation procedures.

Recently, Derman & Kani (1994), Dupire (1994) and Rubinstein (1994) developed a valuation framework in which the local volatility rate is a deterministic, but unspecified, function of asset price and time. If the specification of the volatility function is known, any of the lattice-based or simulation procedures can be applied to value options. Unfortunately, the structural form is not known.[8]

To circumvent this problem, these authors parameterise their model by searching for a binomial or trinomial lattice that achieves an exact cross-sectional fit of reported option prices. An exact cross-sectional fit is always possible because there are as many degrees of freedom in defining the lattice (and, hence, the local volatility rate function) as there are option prices. With the structure of the "implied-tree" identified, it becomes possible to value other, more exotic, OTC options and to refine hedge ratio computations.

The effects of stochastic volatility on option valuation are modelled by either superimposing jumps on the asset price process, allowing volatility to have its own diffusion process, or both. Unfortunately, the introduction of stochastic volatility negates the BSM risk-free hedge argument because volatility movements cannot be hedged. An exception to this rule is Merton (1976), who adds a jump term to the usual geometric Brownian motion governing asset price dynamics. By assuming that the jump component of an asset's return is unsystematic, Merton can create a risk-free portfolio in the BSM sense and apply risk-neutral valuation. Indeed, for European-style options, he finds analytical valuation formulas. If the jump risk is systematic, however, the BSM risk-free hedge cannot be formed, and option valuation will be utility-dependent.

A number of authors model asset price and asset price volatility as separate, but correlated, diffusion processes. Asset price is usually assumed to follow geometric Brownian motion. The assumptions governing volatility vary. Hull & White (1987), for example, assume volatility follows geometric Brownian motion. Scott (1987) models volatility using a mean-reverting process and Wiggins (1987) uses a general Wiener process. Bates (1996) combines both jump and volatility diffusions in valuing foreign currency options. Except in the uninteresting case where asset price and volatility movements are independent, these models require the estimation of risk premiums.

The problem when volatility is stochastic is that a risk-free hedge cannot be created since volatility is not a traded asset. But, perhaps, this problem is only temporary. Derivatives contracts on volatility have been discussed in a variety of forums[9], and, indeed, an option on the VDax (Volatility Dax) is to be introduced at the Deutsche Terminbörse in January 1998. The critical issue is, of course, the "correct" contract design.

Summary

This article organises and highlights some of the important option valuation research contributions of the past 25 years. All of them build upon the simple, but powerful, risk-free hedge insight in the work of Black & Scholes (1973) and Merton (1973). That the Nobel committee has finally recognised the brilliance of their work is commendable.

Sadly, I am reminded of a single, prophetic statement: "Like many great inventions, it started with some tinkering and ended with delayed recognition." (Fischer Black, *Journal of Portfolio Management*, spring 1989.) ■

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[7] See Cox & Ross (1976)
[8] Dumas, Fleming & Whaley (1998) implement the deterministic volatility function option valuation model by expanding the local volatility rate function in a Taylor series and estimating the parameters of the function directly
[9] See, for example, Whaley (1993)