PART

2

FUTURES

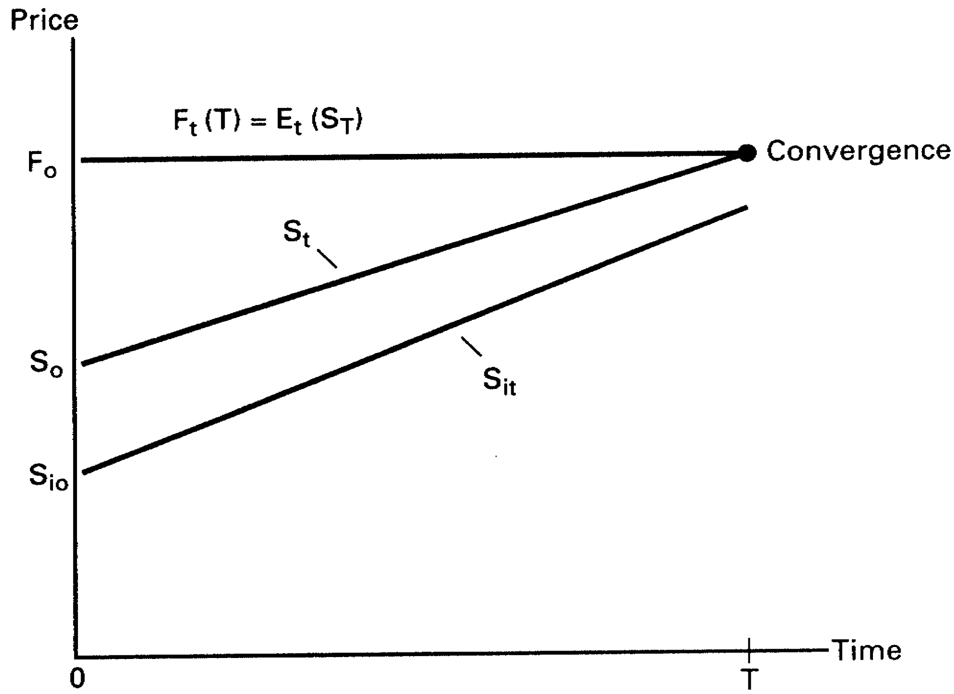
3 STRUCTURE OF FUTURES PRICES

Futures contracts intermediate between the present and the future. Futures prices are observed today but refer to transactions to be carried out in the future. As such, futures prices must reflect expectations about the future. The relation between the futures price and the expected spot price in the future is called the *intertemporal structure of futures prices.* In this chapter, we assume that the futures price on a contract maturing at time T, $F_t(T)$, equals the expected spot price of the underlying asset at time T, $E(\tilde{S}_T)$. The subscript, t, indicates the time at which the futures price is observed and the expectation is formed. In other words, we assume that $F_t(T) = E(\tilde{S}_T)$ for all t, as depicted in Figure 3.1. The relation between the futures price and the expected spot price is investigated in detail in Chapter 5.

Even if the futures price equals the expected spot price on average, the actual paths of the futures price and the spot price are erratic because new information entering the market causes investors to revise expectations and hence prices. Holders and processors of the commodity underlying the futures contract use the futures contract to guard against unanticipated price fluctuations. They hedge their positions in the underlying commodity by taking an offsetting position in the futures contract. The role of hedging and the effectiveness of hedging strategies are examined in Chapter 4.

In this chapter, the focus is on the relation between the current futures and spot prices, represented by the leftmost vertical line segment joining the commodity and futures markets in Figure 1.1. In terms of Figure 3.1, we investigate the relation of observable prices at time t. At any time t, we can observe a spot price for an underlying commodity and several futures prices for different maturity dates. The difference between $F_t(T)$ and S_t shown in Figure 3.1 is that $F_t(T)$ refers to the price

FIGURE 3.1 Structure of Futures Prices



at time t of the underlying commodity to be delivered at time T and S_t is the price at time t of the underlying commodity for immediate delivery. If the underlying commodity is being stored and if it costs something to store it between t and T, the spot price will be less than the futures price as shown in the figure. Sometimes the spot price exceeds the futures price. This occurs when new supplies of the underlying commodity, as in a crop harvest, are expected in the future.

Frequently, futures contracts are written on commodities that are not uniquely definable. For example, there may be different grades of wheat, or wheat may be held in different locations. In other words, there may be many different spot prices, $S_{1t}, S_{2t}, S_{3t}, \cdots S_{nt}$. Because of transportation costs and grade differences, the spot price of a particular grade or at a particular location may be more or less than the spot price of the commodity defined in the futures contract. Sometimes the term *cash price* rather than spot price is used to refer to the price of a particular grade at a particular location, and *spot price* refers to the commodity defined by the futures contract. Processors and holders of the underlying commodity usually deal at the cash price and use the futures market as a way to hedge fluctuations in the cash price. We use S_t to denote the spot price at time t of the commodity defined by the futures contract and S_{it} to denote the cash price of a particular grade or location i at time t.

For some underlying commodities (typically, financial instruments), the distinction between the cash price and the spot price is not important. In the case of a foreign currency, for example, there is only one grade of the underlying commodity and there are no important differences in deliverability by location.

3.1 THE BASIS

The total basis is the difference between the current futures price and the cash price, $F_t - S_{it}$. The total basis contains two components:

$$\text{Total basis} = \text{Time basis} + \text{Space and grade basis}$$

or, algebraically,

$$F_t - S_{it} = (F_t - S_t) + (S_t - S_{it}). \quad \textbf{(3.1)}$$

Since there are many grades and delivery points in some commodities, the space and grade basis, $S_t - S_{it}$, can take on many values depending on transportation costs and grade differences. This component of the total basis is not considered until Chapter 4. We return to it when we consider hedging effectiveness of futures contracts on particular commodities, since the hedging effectiveness of a futures contract depends on how closely futures price movements correspond to price movements in the cash commodity being hedged.

The lack of costless arbitrage opportunities in a rational, frictionless market determines the equilibrium relation between the contemporaneous futures and spot prices, F_t and S_t, and hence the time basis, $F_t - S_t$. Arbitrage depends on the convergence of F_t and S_t at the maturity of the futures contract, T. At maturity, the futures price of the expiring contract must be the same as the spot price, $F_T = S_T$, or a riskless arbitrage opportunity would exist. If the futures price at maturity were less than the spot price at maturity, arbitrageurs would purchase futures contracts, take delivery, and sell the delivered commodity at the spot price. On the other hand, if the futures price at maturity were greater than the spot price at maturity, arbitrageurs would sell futures, buy the spot commodity, and make delivery against the futures contract. These opportunities are avoided only if, at the last moment of the contract's life, the futures price equals the spot price of the underlying commodity.

At times prior to maturity, a similar arbitrage is possible. If the underlying commodity is available, one can purchase the commodity at time t at the price S_t and hold it until maturity for delivery against the futures contract. Such a strategy is profitable if F_t exceeds S_t by more than the cost of carrying the commodity to maturity. In such a case, arbitrageurs would sell futures contracts at F_t, buy the underlying commodity at S_t, and carry the commodity to maturity for delivery against the futures contract. If such costless arbitrage opportunities do not exist, the relation between the futures price and the spot price is

$$F_t \leq S_t + B, \quad \textbf{(3.2)}$$

where B is the cost of carrying the commodity until maturity of the futures contract.

The relation (3.2) limits the amount by which the futures price can exceed the spot price. This limit results from the fact that it is always possible to acquire the commodity for future delivery by buying it today and holding it rather than by purchasing a futures contract.

For certain commodities such as financial instruments, arbitrageurs can make profits if the futures price falls too low relative to the spot price by engaging in reverse arbitrage, namely selling the spot commodity and purchasing a futures contract. Reverse arbitrage is possible only if sufficient supplies of the underlying commodity are available. The underlying commodity must be sold by its owner who can replace it with a futures position or it must be loaned to someone else who sells it and buys futures. The sale of the loaned commodity is a *short sale*. The short seller has the obligation to return the commodity on demand. In agricultural commodities, reverse arbitrage is not possible during that part of the crop year in which the commodity is used up or nearly used up. At such times, few people are willing to sell the underlying commodity or lend it out to short sellers. In such cases, the futures price may fall far below the spot price, as a new crop is anticipated. There is no way to make arbitrage profits, however, since there is not enough current supply of the old crop that can be sold to drive down the current spot price.

In most financial instruments, reverse arbitrage is not a problem because a large stock of the cash commodity is on hand. For example, suppose the futures price of the British pound were too low relative to its spot price. Arbitrageurs would purchase futures and short sell the underlying commodity, the pound. When reverse arbitrage is possible, equilibrium requires

$$F_t = S_t + B. \tag{3.3}$$

The relation (3.3) holds for agricultural commodities only while the commodity is being stored or "carried." Consequently, the relation is often called the *cost of carry relation*. Unless specified otherwise, we shall assume a carrying charge market in which the equality (3.3) holds.

3.2 BASIS ARBITRAGE IN DETAIL

The exact form of the cost of carry relation (3.3) depends on how *B* is expressed, how profits and losses accrue on the futures contract, when storage costs are paid, and the level of transaction costs. We turn now to a detailed examination of basis arbitrage in which these issues are considered and the arbitrage is explained more precisely. Throughout our discussion, we assume that the transaction costs of buying and selling the spot commodity and the futures contract are zero.

Carrying Costs

The cost of carrying a commodity usually consists of two components—interest and storage. The interest cost is common to all commodities. If a commodity is held in inventory, the opportunity cost of the funds tied up in holding the inventory is incurred. Over and above interest, however, there may be storage costs. For the agricultural commodities, these costs include warehouse rent, insurance, and spoilage. On the other hand, for some commodities, storage costs are zero or negative. For example, T-bills have negligible storage costs, and T-bonds and stock index portfolios have storage costs that are negative in the sense that coupon yield or dividend yield accrues to holders of these assets.

In this section, we assume that the basis arbitrageur sets aside just enough money to pay for storage costs over the life of the position. If the interest rate and storage costs are known, the amount of money or, alternatively, the size of the "storage cost fund" can always be established today. If interest rates and storage costs were not known with certainty, the appropriate size of the fund would be unclear; and, as a result, the basis arbitrage would have some risk arising from this uncertainty. The exact size of the storage cost fund depends on when and how storage costs are paid.

Storage Costs Paid at Maturity. A simple assumption is that storage costs, $B - r^*S_t$, are paid at maturity, where r^* is the riskless rate of interest over the life of the futures contract. Table 3.1 illustrates the basis arbitrage that, under this assumption, leads to relation (3.3). At time t, the arbitrageur buys one unit of the spot commodity for S_t. When the futures contract matures at time T, the commodity value is $\tilde{S}_T$ and the storage costs, $B - r^*S_t$, are paid. To account for the impending storage cost payment, an amount of money, $(B - r^*S_t)/(1 + r^*)$, is set aside in a storage cost fund. This amount is simply the present value of the storage costs to be paid at time T. The fund is invested at the certain rate of interest, r^*, to guarantee that the necessary amount is available at maturity. The arbitrageur borrows enough to cover the initial cost of the spot commodity and the storage fund. Finally, the arbitrageur takes a short futures position. No cash payment is required in order to establish a futures position so long as collateral to guarantee adherence to the futures contract is deposited. We assume that the spot commodity position provides such collateral. The terminal value associated with each of these initial transactions is shown in the last column of the table. The uncertain commodity price is $\tilde{S}_T$, the necessary storage cost payment is exactly covered by the balance of the storage fund, the face amount of the loan to be repaid at maturity is $S_t(1 + r^*) + (B - r^*S_t) = S_t + B$, and the uncertain value of the short futures position is $F_t - \tilde{S}_T$.

The initial cost of setting up the arbitrage position is zero, as indicated by the sum of the initial value column in Table 3.1. Since the net terminal value is certain, the absence of costless arbitrage opportunities in a rationally functioning market requires that the sum of the terminal value column, $F_t - S_t - B$, also be zero. If the net terminal value were positive, arbitrageurs would store the cash commodity as the table indicates and make a riskless profit with no initial investment. If the net terminal value in Table 3.1 were negative, arbitrageurs would try to reverse the transactions shown in the table in order to make a riskless profit with no initial investment. Reverse arbitrage is possible in financial futures such as T-bonds and stock index futures. In such cases, arbitrageurs would short sell the financial instrument, invest the proceeds at the riskless rate, and buy futures. These arbitrage forces bring about the equilibrium price relation, $F_t = S_t + B$.

TABLE 3.1 Arbitrage transactions for establishing the relation between futures and spot prices, $F_t = S_t + B$.

Position	Initial Value	Terminal Value
Buy one unit of spot commodity	$-S_t$	$\tilde{S}_T - (B - r^* S_t)$
Create storage fund	$-(B - r^* S_t)/(1 + r^*)$	$+(B - r^* S_t)$
Borrow	$S_t + (B - r^* S_t)/(1 + r^*)$	$-(S_t + B)$
Sell futures contract	0	$F_t - \tilde{S}_T$
Net portfolio value	0	$F_t - S_t - B$

The arbitrage presented in Table 3.1 underlies the cost of carry relation (3.3). In subsequent chapters, we also rely on a slightly different form of the cost of carry relation,

$$F_t = S_t(1 + b^*), \tag{3.4}$$

where b^* is the cost of carrying the underlying commodity expressed as a proportion of the commodity price, $b^* = B/S_t$. This rate (like the riskless interest rate, r^*) corresponds to the life of the futures contract. The arbitrage underlying this relation is shown in Table 3.2. Note that we have dropped the direct investment in the storage fund at time t as well as the borrowings to set up the fund, since they are offsetting transactions. At time T, the commodity is delivered against the short futures position, eliminating the presence of the $\tilde{S}_T$ terms and the risk of the arbitrage position. The interest terms $S_t r^*$ also cancel. The terminal value of the position is $F_t - S_t(1 + b^*)$, which must be zero in the absence of costless arbitrage opportunities.

Storage and Interest Costs Incurred Continuously. We shall frequently assume that: (a) individuals can borrow or lend risklessly at a compounded rate of interest, r; and (b) the costs of carrying the commodity underlying the futures contract are paid out at a known, continuously compounded rate, b. The interest cost of carrying the commodity is included in b. Both the cost of carry rate, b, and the interest rate, r, are now expressed per unit of time (e.g., annual rates) as opposed to rates over the life of the futures contract to facilitate handling cash flows at different points in time. The use of continuously compounded interest and storage costs simplifies certain arbitrage relations that we examine in this book and takes into account the fact that such costs are often paid during the storage period, not just at maturity.

Under the assumption that storage costs are paid at the continuous rate, b, the appropriate size of the storage cost fund at time zero is $S_0 e^{(b-r)T} - S_0$. We assume the fund is invested in the underlying commodity being stored. As storage costs (for warehouse rent, insurance, and so forth) are incurred, some of the commodity

TABLE 3.2 Arbitrage transactions for establishing the relation between futures and spot prices, $F_t = S_t(1 + b^*)$.

Position	Initial Value	Terminal Value
Buy one unit of spot commodity	$-S_t$	$\tilde{S}_T - S_t(b^* - r^*)$
Borrow S_t	S_t	$-S_t(1 + r^*)$
Sell futures contract	0	$-(\tilde{S}_T - F_t)$
Net portfolio value	0	$F_t - S_t(1 + b^*)$

is sold to pay for these costs. A reduction in the amount of the commodity also occurs if the commodity deteriorates while in storage. The storage cost fund is not used to pay the interest costs of holding the commodity and therefore the term $b - r$, representing carrying costs net of interest, appears in the expression for the fund. Interest costs are treated separately and are assumed to accrue continuously and be paid at time T. For example, suppose that b is 0.0006 per day, r is 0.0002 per day, 100 days remain to maturity, and the spot price is 50. The current value of the storage cost fund is $S_0(e^{(b-r)T} - 1) = 50(e^{(.0006-.0002)100} - 1) = 2.04$.

The initial fund may be negative if the underlying commodity has an income yield. For example, if the underlying commodity is a stock index portfolio that pays a constant dividend rate, d, the cost of carry rate is $b = r - d < r$, which causes the storage fund to be negative. Each day the fund grows to reflect the dividend yield that accrues over the day. Suppose that $b = r - d = 0.0002 - 0.0001 = 0.0001$ per day and other features of the example remain the same. Then the initial fund is $50(e^{.01-.02} - 1) = -0.50$. A negative storage fund is a short position in the underlying commodity that is covered as dividend payments are received. A detailed examination of how the fund changes through time is contained in Appendix 3.1.

Table 3.3 illustrates the basis arbitrage when storage costs are paid continuously. When the portfolio is formed, the net investment cost equals zero, since the cost of acquiring the commodity is completely financed with riskless borrowing. As a result, the futures position in Table 3.3 requires zero initial outlay and has a zero initial value. At the expiration of the futures contract, the one unit of the commodity on hand is delivered against the short futures position, and the net terminal value of the portfolio is $F_t - S_te^{b(T-t)}$. This term cannot be positive, otherwise costless arbitrage profits would be possible. And, if $F_t - S_te^{b(T-t)} \leq 0$,

$$F_t \leq S_te^{b(T-t)}. \tag{3.5}$$

The relation (3.5) limits the amount by which the futures price can exceed the spot price. Again, this limit results from the fact that it is always possible to

TABLE 3.3 Arbitrage transactions for establishing the relation between futures and spot prices, $F_t \leq S_t e^{b(T-t)}$.

Position	Initial Value	Terminal Value
Buy one unit of commodity in spot market	$-S_t$	$\tilde{S}_T$
Create storage fund[a]	$-[S_t e^{(b-r)(T-t)} - S_t]$	0
Borrow $S_t e^{(b-r)(T-t)}$	$S_t e^{(b-r)(T-t)}$	$-S_t e^{b(T-t)}$
Sell futures contract	0	$F_t - \tilde{S}_T$
Net portfolio value	0	$F_t - S_t e^{b(T-t)}$

a. Carrying costs are assumed to be paid in units of the commodity each day during the arbitrage. The storage fund has $e^{(b-r)(T-t)} - 1$ units on hand on day t and is reduced on each subsequent day. In general, the number of units of the commodity in the fund on day τ is $e^{(b-r)(T-\tau)} - 1$, so the number of units on day τ is $e^{(b-r)(T-T)} - 1 = 0$.

acquire the commodity for future delivery by buying it today and holding it rather than by buying a futures contract. When reverse arbitrage is possible, as in the case of financial futures, the equilibrium price relation is

$$F_t = S_t e^{b(T-t)}. \tag{3.6}$$

While the basis is frequently defined as the absolute dollar difference between the futures price and the spot price, it is often preferable to define the basis in proportional terms as F_t/S_t. Under this definition, the basis does not depend on the absolute price level of the commodity, and comparisons across commodities are possible. We see from equation (3.6) that the proportional basis must equal the proportional carrying cost in equilibrium:

$$\frac{F_t}{S_t} = e^{b(T-t)}. \tag{3.7}$$

The basis and the basis behavior are examined in greater detail when individual commodities are considered in Chapters 6 through 9.

3.3 TERM STRUCTURE OF FUTURES PRICES

At any time, several futures contracts with different times to maturity are outstanding. The relation between the futures prices and the time to maturity is called the *term structure of futures prices*. The relation between two futures prices is determined by the same factors that determine the relation between the futures price and the spot price. That makes sense because we are simply looking farther out in time. Let $F_t(T_1)$ be the time t futures price on a contract maturing at time T_1, and $F_t(T_2)$ be the time t futures price on a contract maturing at time T_2, where $T_1 < T_2$.[1] The lack of costless arbitrage opportunities in the futures market ensures that

$$F_t(T_2) \leq F_t(T_1)e^{b(T_2-T_1)}. \tag{3.8}$$

If $F_t(T_2)$ exceeds the value specified by (3.8), arbitrageurs would sell the distant futures at $F_t(T_2)$ and buy near-term futures at $F_t(T_1)$, take delivery at time T_1, and hold the commodity until T_2. If short selling of the spot commodity is possible, condition (3.8) would hold as the equality,

$$F_t(T_2) = F_t(T_1)e^{b(T_2-T_1)}. \tag{3.9}$$

Reverse arbitrage would maintain the equality by the following strategy: If $F_t(T_2)$ were too low, arbitrageurs would buy futures at $F_t(T_2)$ and sell at $F_t(T_1)$. At time T_1, the arbitrageurs would borrow the underlying commodity and deliver it against the futures contract.

The difference between the futures prices of two maturities is called a *spread.* Individuals who buy one maturity and sell another maturity are called *spreaders.* Spreaders usually liquidate their position before delivery is required. They hope that the basis relation between the two maturities will be re-established before the expiration of the "short leg" of the spread.

3.4 FORWARD AND FUTURES CONTRACTS

In Chapter 2 we saw that futures contracts require daily settlement of profits and losses. A forward contract is identical to a futures contract in all respects, except that, with a forward contract, profits and losses are realized only at maturity or when the forward position is reversed.[2] Thus, it is possible for the buyer of a futures contract to suffer short term losses (due to decline in the futures price) even if the contract is not liquidated, while the buyer of a forward contract would not incur those same losses unless he liquidates his forward contract position. The difference between futures contracts and forward contracts lies in the fact that gains (losses) can be invested (financed) at the short-term interest rate, while gains or losses on forward contracts are not recognized until the contract matures or is liquidated. At maturity, the futures and forward contracts have claims on the same amount of the commodity, so that the differences between these contracts have to do with the timing of gains and losses.

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[1]In general, we talk about a single futures contract written on a single underlying commodity, so we suppress the contract maturity notation in parentheses.

[2]In this section, we assume that there is no difficulty or cost in reversing a forward position before maturity.

The difference in the pattern of cash flows of the forward and futures positions means that the value of a forward contract position is slightly different from the value of a futures contract position. This difference is explained in detail in the first part of Appendix 3.2. However, if the short-term interest rates are constant, the price of a forward contract equals the price of the futures contract, as the second part of Appendix 3.2 shows. The difference between the value of a futures position and a forward position is reflected in a difference in the number of units of the commodity held and not in a difference between the futures and forward prices. Appendix 3.2 also shows how the number of units of a futures contract can be adjusted to make the terminal values of the forward and futures positions equal.

For expositional purposes, we shall assume that a futures contract is the same as a forward contract. While the two are equivalent only if interest rates are known and appropriate adjustments are made in contract size (as shown in Appendix 3.2), the assumption simplifies the presentation considerably.

3.5 SUMMARY

In this chapter, the arbitrage process that links the spot price and futures price is described. Since the underlying commodity can be purchased and held for future delivery, the futures price cannot exceed the spot price by more than the cost of carrying the commodity to the maturity of the futures contract. Similarly, so long as supplies of the commodity exist, the futures price cannot fall below the cost of carry equilibrium price; for if it did, arbitrageurs would short-sell the spot commodity and buy futures. When storage and interest costs are incurred continuously, the relation of the futures and spot prices is given as

$$F_t = S_t e^{b(T-t)}, \tag{3.10}$$

where b is the cost of carrying the commodity, including interest, $T - t$ is the time to maturity of the futures contract, and F_t and S_t are the futures and spot prices at time t, respectively. The term structure of futures prices is defined, and the difference between futures and forward contracts is explained.

APPENDIX 3.1

THE STORAGE COST FUND

In this chapter, as well as many of the futures and options chapters to follow, we use the concept of a storage cost fund. This fund is established to (a) pay the costs (other than interest) of carrying a commodity over the storage period or (b) receive any income earned on the underlying commodity. The fund is invested in the underlying commodity being stored. Each day the number of units held in the storage cost fund is changed slightly to reflect that day's storage cost or income. In the basis arbitrage context, the storage cost fund begins at the end of day 0 with an investment of $e^{(b-r)T} - 1$ units. During each subsequent day, the commodity position is reduced (increased) by the factor $e^{-(b-r)}$ if $b > r$ ($b < r$). The reduction in the commodity position, assuming $b > r$, reflects the payment of carrying costs other than interest. In the case of an agricultural commodity, for example, warehouse rent, insurance, or natural deterioration of the stored commodity must be paid while the commodity is held. On the other hand, if the underlying commodity is, say, a stock index portfolio that pays a dividend so that $b < r$, the storage fund, which starts as a negative amount (a short position in the commodity), is increased each day to reflect the dividend yield that accrued over the day. Using such a scheme ensures that the storage cost fund is zero and exactly one unit of the commodity is held when the futures contract expires at time T, as is shown in Table 3.1a.

We refer to the sum of the commodity position and the position in the storage fund as the *rollover position* in the commodity. The number of units in the rollover position changes as the commodity is sold off to pay storage costs.

To clarify the storage cost fund concept, consider a case where the cost of carry rate, b, is 0.01 per day, the riskless rate of interest, r, is 0.005, and the holding period is ten days. The amount in the storage cost fund at the end of day 0 is therefore $e^{(.01-.005)10} - 1 = 0.051271$ units, as is shown in Table 3.1b; and the number of units in the rollover position is 1.051271. If the commodity price is 100, the total value of the rollover position is 105.1271. On day 1, the total number of units in the commodity is reduced from 1.051271 by the factor $e^{-(.01-.005)} = 0.995012$. Thus, the total number of units of the commodity remaining is $1.051271 \times 0.995012 = 1.046028$, with a total value of $97 \times 1.046028 = 101.4647$. Day after day, the number of units is reduced. With the cost of carry rate, the interest rate, and the holding period known, exactly one unit of the commodity remains on hand at the end of day T.

TABLE 3.1a End-of-day positions in the commodity and the storage cost fund required to guarantee one unit of the commodity on hand at maturity assuming constant cost of carry and interest rates, b and r.

Day	Units of Commodity	Units of Commodity in Storage Fund[a]	Total Value of Commodity Position
0	1	$e^{(b-r)T} - 1$	$S_0 e^{(b-r)T}$
1	1	$e^{(b-r)(T-1)} - 1$	$\tilde{S}_1 e^{(b-r)(T-1)}$
2	1	$e^{(b-r)(T-2)} - 1$	$\tilde{S}_2 e^{(b-r)(T-2)}$
⋮	⋮	⋮	⋮
t	1	$e^{(b-r)(T-t)} - 1$	$\tilde{S}_t e^{(b-r)(T-t)}$
⋮	⋮	⋮	⋮
T	1	0	$\tilde{S}_T$

a. Each day, storage costs are assessed by reducing the total number of units of the commodity by the factor $e^{-(b-r)}$. Thus, at the end of day 1, the total number of units on hand equals the total number of units on hand on day 0, $e^{(b-r)T}$, times the factor, $e^{-(b-r)}$, or $e^{(b-r)(T-1)}$, and so on.

TABLE 3.1b End-of-day positions in the commodity and the storage cost fund required to guarantee one unit of the commodity on hand at maturity assuming constant daily cost of carry and interest rates, $b = .01$ and $r = .005$, respectively, and a holding period of $T = 10$ days.

Day	Units of Commodity	Units of Commodity in Storage Fund[a]	Commodity Price	Total Value of Commodity Position
0	1	0.051271	100	105.1271
1	1	0.046028	97	101.4647
2	1	0.040811	95	98.8770
3	1	0.035620	98	101.4907
4	1	0.030455	102	105.1064
5	1	0.025315	99	101.5062
6	1	0.020201	101	103.0403
7	1	0.015113	104	105.5718
8	1	0.010050	106	107.0653
9	1	0.005013	105	105.5263
10	1	0	107	107.0000

a. Each day, storage costs are assessed by reducing the total number of units of the commodity by the factor $e^{-(.01-.005)}$. Thus, at the end of day 1, the total number of units on hand equals the total number of units on hand on day 0, 1.051271, times the factor, 0.995012, or 1.046028, and so on.

APPENDIX 3.2

FORWARD AND FUTURES CONTRACTS[1]

The Difference Between Forward and Futures Positions

To understand the distinction between a long position in a forward contract and a long position in a futures contract, we need to establish the value of the contract positions at some arbitrary point in time t prior to the contracts' maturity, $t < T$.[2] Let f_t be the price at time t of a forward contract maturing at time T, and consider the following course of events. Suppose an individual buys a forward contract at time 0 and lets some time elapse. Now, suppose that the individual unwinds her position at time t (where $t < T$) by selling a different forward contract that also matures at time T. These transactions are summarized in Table 3.2a. Note that the two transactions produce a certain net terminal value (at time T) equal to $f_t - f_0$. The time t value of the forward position established at time 0 must therefore equal the present value of the terminal amount $f_t - f_0$, that is,

$$(f_t - f_0)e^{-r(T-t)}, \qquad \textbf{(A3.1)}$$

otherwise costless arbitrage opportunities would exist. That is, the buyer of the forward contract at time t must be indifferent between a new contract priced at f_t for which no payment is required and the existing contract for which payment $(f_t - f_0)e^{-r(T-t)}$ is necessary. At maturity, the difference $f_t - f_0$ will be realized. The value at time t is just the present value of that amount.

The value at time t of a futures contract position established at time 0 is the sum of the daily gains and losses on the futures position carried forward to t at interest rate r, that is,

$$\sum_{\tau=1}^{t}(F_\tau - F_{\tau-1})e^{r(t-\tau)} = \sum_{\tau=1}^{t}(F_\tau - F_{\tau-1})e^{-r(T-t)}e^{r(T-\tau)}, \qquad \textbf{(A3.2)}$$

as is shown in Table 3.2b. Each day the gain or loss is posted and carried forward to t. The futures contract is then rewritten at the new futures price.

The expressions (A3.1) and (A3.2) reveal exactly why a long forward position is different from a long futures position. The difference arises because the daily gains or losses on the futures position are taken forward at the riskless interest rate,

[1] The analysis in this section is drawn from Stoll and Whaley (1986), pp. 25–62.

[2] We assume that both positions were formed at time 0, where $0 < t$, and that the forward and futures contracts are written on the same underlying commodity and have the same time to expiration.

TABLE 3.2a Transactions for establishing the value of an existing forward contract position.

Position	Initial Value (0)	Intermediate Value (t)	Terminal Value (T)
Buy forward contract at time 0	0		$\tilde{S}_T - f_0$
Sell forward contract at time t		0	$-(\tilde{S}_T - f_t)$
Net forward position			$f_t - f_0$

while the daily gains or losses on the forward contract are not recognized and therefore are implicitly carried forward with no interest. At any point in time after the futures and forward positions are entered, the value of a futures position will, in general, be different from the value of a forward position, unless, of course, the interest rate is zero. If futures and forward prices have increased on average since the positions were taken, the value of a long futures position will exceed the value of a long forward position because of accumulated interest. On the other hand, if futures and forward prices have decreased on average since the positions were taken, the value of the forward position will be greater.

Forward and Futures Prices Are Equal If the Interest Rate Is Known

Although the values of the two contract positions differ, the prices of the forward and futures contracts are equal (i.e., $f = F$) if the gains and losses on the futures

TABLE 3.2b Value of an existing futures contract, assuming constant interest rate, r.

Day	Futures Price	Daily Profit	Profit at Time t
0	F_0		
1	$\tilde{F}_1$	$(\tilde{F}_1 - F_0)$	$(\tilde{F}_1 - F_0)e^{r(t-1)}$
2	$\tilde{F}_2$	$(\tilde{F}_2 - \tilde{F}_1)$	$(\tilde{F}_2 - \tilde{F}_1)e^{r(t-2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	$\tilde{F}_t$	$\tilde{F}_t - \tilde{F}_{t-1}$	$\tilde{F}_t - \tilde{F}_{t-1}$
Total value at time t			$\sum_{\tau=1}^{t}(F_\tau - F_{\tau-1})e^{r(t-\tau)}$

TABLE 3.2c Terminal value of a long rollover position in futures, assuming constant interest rate, r.

Day	No. of Contracts at End of Day[a]	Futures Price at End of Day	Daily Profit	Accumulation Factor	Daily Profit at Maturity
0	e^{-rT}	F_0			
1	$e^{-r(T-1)}$	$\tilde{F}_1$	$e^{-r(T-1)}(\tilde{F}_1 - F_0)$	$e^{r(T-1)}$	$\tilde{F}_1 - F_0$
2	$e^{-r(T-2)}$	$\tilde{F}_2$	$e^{-r(T-2)}(\tilde{F}_2 - \tilde{F}_1)$	$e^{r(T-2)}$	$\tilde{F}_2 - \tilde{F}_1$
⋮	⋮	⋮	⋮	⋮	⋮
t	$e^{-r(T-t)}$	$\tilde{F}_t$	$e^{-r(T-t)}(\tilde{F}_t - \tilde{F}_{t-1})$	$e^{r(T-t)}$	$\tilde{F}_t - \tilde{F}_{t-1}$
⋮	⋮	⋮	⋮	⋮	⋮
T	1	$\tilde{F}_T$	$\tilde{F}_T - \tilde{F}_{T-1}$	1	$\tilde{F}_T - \tilde{F}_{T-1}$
	Value of rollover futures position at maturity				$\tilde{F}_T - F_0$

a. Each day, the number of futures contracts is increased by the factor e^r. Thus, at the end of day 1, the total number of futures contracts equals the total number of futures contracts on day 0, e^{-rT}, times the factor, e^r, or $e^{-r(T-1)}$, and so on.

position accumulate interest at *known* rates.[3] Here, we invoke a stronger assumption than is necessary to demonstrate the equivalence of the forward and futures prices. We assume that the short-term interest rate is constant during the contract lives. To see that the futures price equals the forward price, we establish a portfolio that consists of a long rollover position in the futures and a short position in the corresponding forward.[4] The rollover position adjusts for the daily profits and losses in the futures contract and is shown in Table 3.2c. The rollover position in the futures begins at the end of day 0 with e^{-rT} futures contracts purchased. On each subsequent day, the futures profits/losses are recognized, and the number of futures contracts is increased by a factor of e^r. On the last day of the futures' life, exactly one futures contract is held. Assuming the daily gains or losses are invested or financed at the riskless rate of interest r, the terminal value of the long rollover position in the futures equals $\tilde{F}_T - F_0$, as is shown in Table 3.2c.

To illustrate the concept of a rollover futures position, consider Table 3.2d. In the table, the riskless rate of interest is assumed to be 0.005 per day. At the end

[3]See Cox, Ingersoll, and Ross (1981).

[4]Naturally, the futures and forward contracts used in this analysis are written on the same underlying spot commodity and have the same maturity date.

TABLE 3.2d Terminal value of a long rollover position in futures, assuming constant daily interest rate of $r = 0.005$ and a holding period of $T = 10$ days.

Day	No. of Contracts at End of Day[a]	Futures Price at End of Day	Daily Profit	Accumulation Factor	Daily Profit at Maturity
0	0.951229	80			
1	0.955997	83	2.867992	1.046028	3
2	0.960789	84	0.960789	1.040811	1
3	0.965605	79	–4.828027	1.035620	–5
4	0.970446	81	1.940891	1.030455	2
5	0.975310	83	1.950620	1.025315	2
6	0.980199	85	1.960397	1.020201	2
7	0.985112	84	–0.985112	1.015113	–1
8	0.990050	87	2.970150	1.010050	3
9	0.995012	88	0.995012	1.005013	1
10	1	90	2	1	2
Value of rollover futures position at maturity					10

a. Each day, the number of futures contracts is increased by the factor $e^{.005}$. Thus, at the end of day 1, the total number of futures contracts equals the total number of futures contracts on day 0, 0.951229, times the factor, 1.005012, or 0.955997, and so on.

of day 0, $e^{-.005(10)} = 0.951229$ futures contracts are purchased. At the end of day 1, the position is increased by a factor of $e^{.005} = 1.005012$, so the number of contracts held is 0.955997, and so on. Note that the profits/losses from the rollover position each day correspond exactly to the gain/loss on the futures for that day, once the profit/loss is taken forward to time T. At time T, the daily profits/losses plus all interest accumulated/paid on the daily profits/losses sum exactly to the futures price change over the ten-day period, 10. Note that the rollover position in futures is different from the rollover position in the commodity presented in Tables 3.1a and 3.1b. The futures rollover position adjusts for the daily settlement cash flows, while the commodity rollover position adjusts for the daily storage cost payments.

Table 3.2e shows the initial and terminal values of the portfolio consisting of the long rollover futures position and the short forward contract. The initial values are both assumed to be zero since no margin deposit is required. At the expiration of the contracts, both the terminal price of the futures, $\tilde{F}_T$, and the terminal price of the forward, $\tilde{f}_T$, must equal the underlying commodity price, $\tilde{S}_T$, so the net terminal value of the portfolio is $f_0 - F_0$. This value is certain and requires no investment outlay. In equilibrium, the only allowable outcome for a riskless, costless

TABLE 3.2e Arbitrage transactions for establishing the equivalence of forward and futures prices, $f_0 = F_0$.

Position	Initial Value	Terminal Value
Buy rollover position in futures	0	$\tilde{F}_T - F_0 = \tilde{S}_T - F_0$
Sell forward contract	0	$-(\tilde{f}_T - f_0) = -(\tilde{S}_T - f_0)$
Net portfolio value	0	$f_0 - F_0$

investment portfolio is zero, otherwise costless arbitrage profits can be earned. If the net portfolio value is zero, the futures price must equal the forward price, that is, $f_0 = F_0$, and, since the initial time 0 can be any time, $f_t = F_t$ for all t.

The above arbitrage argument showing the equality of forward and futures prices requires a known riskless rate of interest and the implementation of a rollover strategy. In practice, perfect equality of forward and futures prices may not be possible because interest rates are uncertain and because the rollover futures strategy is difficult to implement.