11 VALUATION OF OPTIONS

In the last chapter, we examined the structure of option prices implied by the absence of arbitrage opportunities. The approach in that chapter provided many interesting pricing relations, but the results took the form of option pricing bounds rather than valuation equations. In this chapter, we develop valuation equations for European commodity options by invoking an assumption that commodity prices are lognormally distributed at the option's expiration.

The approach used here also assumes that options are valued as though all individuals in the economy are risk-neutral. This assumption is reasonable because the value of the option does not depend on the expected rate of return of the underlying commodity. The concept of risk-neutral valuation of options and its equivalence to risk-averse valuation are explained in section 1. Section 2 examines the implications of the assumption that commodity prices are lognormally distributed. The assumptions of lognormality and risk-neutrality are then used to price a European call option in the third section and a European put option in the fourth section. Section 5 describes the sensitivity of option price to changes in the option's underlying determinants. Section 6 presents the valuation equation for an option that permits its holder to exchange one risky commodity for another. This option, called an exchange option, is embedded in many types of futures contracts. Valuation approximation methods for American options are briefly described in section 7. Section 8 describes how the parameters of the valuation equations can be estimated, and section 9 concludes with a brief summary.

11.1 RISK-NEUTRAL VALUATION

The value of an option at maturity depends on the value of the underlying commodity. Before maturity, one can calculate the expected value of an option based on the probability distribution of the terminal value of the commodity underlying the option. A probability distribution is given in Figure 11.1, and an option exercise value, X, is shown. The expected value of a call option at maturity is the profit from the call times its probability, summed over all possible values of the underlying commodity. Since the value of the call is zero to the left of X, the expected value of the call is the partial expectation to the right of X. To illustrate this pricing principle, suppose, instead of the smooth probability distribution shown in Figure 11.1, the underlying commodity can only take on the values, 80, 90, 100, 110, and 120, with corresponding probabilities of 0.15, 0.20, 0.30, 0.20, and 0.15. Also, suppose that the exercise price of the call is 100. The expected value of a call is, therefore, $(110 - 100).20 + (120 - 100).15 = 5.0$.

The current value of the call is the discounted value of its expected value at maturity. Determining this present value is a perplexing problem. Under the traditional approach of risk-averse individuals in the economy, the current value of the call is computed by discounting the expected value of the call at expiration at the risk-adjusted rate of return of the call. This is the approach derived by Samuelson in 1965. Unfortunately, his approach requires the estimation of both the expected rate of return on the commodity and the expected rate of return on the call. In practice, reliable estimation of these parameters is extremely difficult.

For many years, option valuation could not overcome this difficulty. A breakthrough came in 1973 with a paper by Black and Scholes. They showed that one could establish a riskless hedge between a stock option and the underlying stock,

FIGURE 11.1 Commodity Price Distribution at Time T

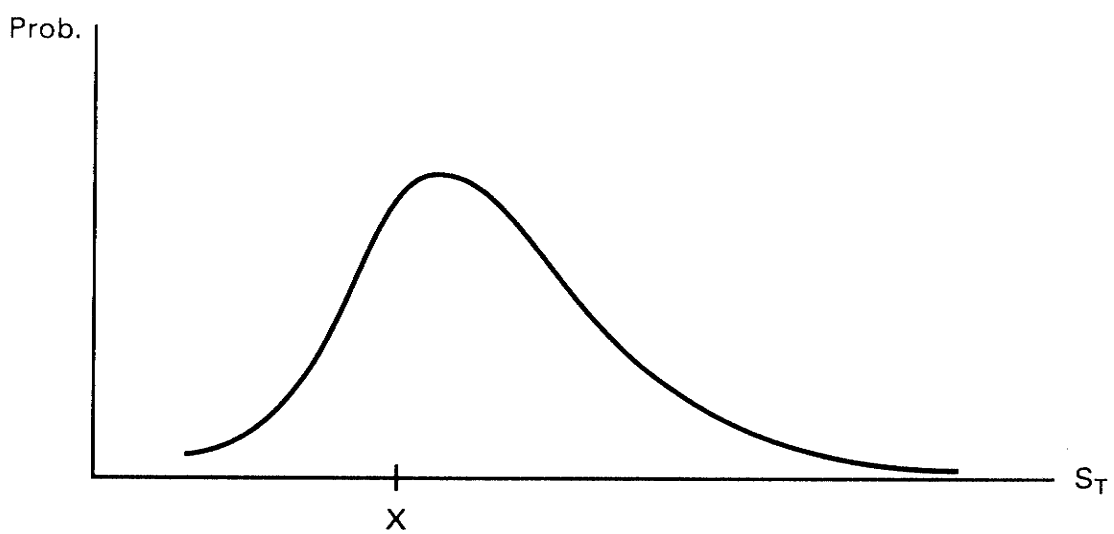
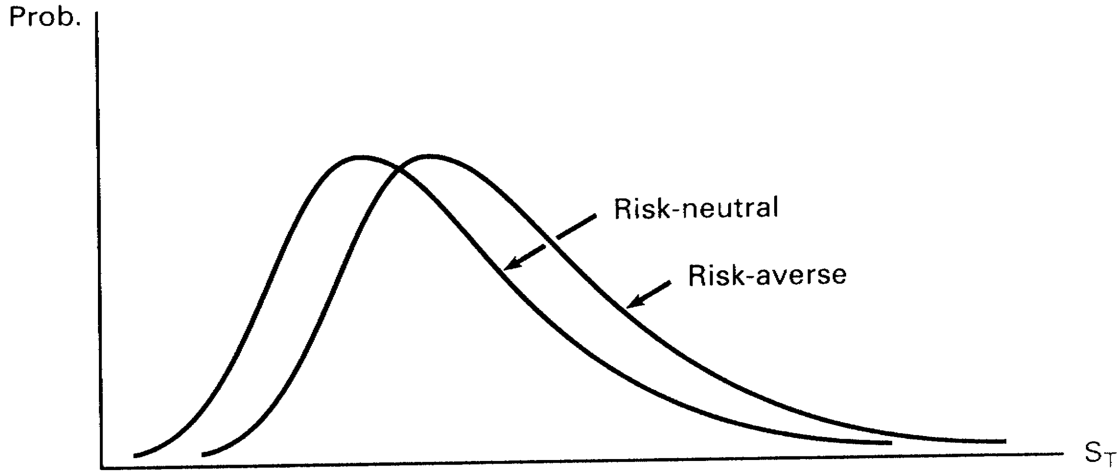


FIGURE 11.2 Commodity Price Distributions for Risk-Neutral and Risk-Averse Individuals



so the option is riskless relative to the stock.[1] Cox and Ross (1976) further showed that they would get the correct option value if they assumed the expected return on the stock and the expected rate of return on the call are the riskless rate, as long as the probability distribution of the ending stock value is otherwise maintained. To illustrate, consider the two probability distributions plotted in Figure 11.2. The probability distribution on the right is the distribution implied if individuals are risk-averse, and the probability distribution on the left is the distribution implied if individuals are risk-neutral. The variances of the two distributions are the same, but the expected value of the risk-neutral distribution is less than the expected value of the risk-averse distribution. Under the Cox-Ross approach, the expected value of the risk-neutral distribution is discounted at the riskless rate of interest, and under the Samuelson approach, the expected value of risk-averse distribution is discounted at a risk-adjusted rate of return. In the end, both approaches provide the same current value for the call.

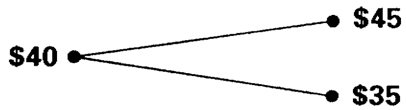
In this chapter, we use the risk-neutral valuation approach because of its mathematical tractability. Prior to doing so, however, we will demonstrate through an illustration using a simple binomial model that the two approaches produce the same result. First, we demonstrate the concept of a riskless hedge. Second, we show risk-neutral valuation. Finally, we show the equivalence of risk-averse valuation to risk-neutral valuation.

Riskless Hedge Portfolio Using a Simple Binomial Model

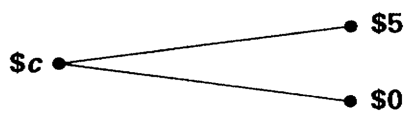
The key insight in the derivation of the option pricing formulas presented in this chapter is that a riskless hedge may be formed between the option and the underlying commodity. To understand the riskless hedge concept, consider the following simple numerical problem. Suppose that the current commodity price is $40 and

[1] For an historical recount of the development of the Black–Scholes option pricing model, see Black (1989).

that at the end of three months the commodity price will be \$45 or \$35. The figure below illustrates the possible commodity price movements.



Now, consider a European call option written on this commodity. This call has an exercise price of \$40 and expires in exactly three months. At expiration, this call will have a value of \$5 or \$0, depending on whether the commodity price is \$45 or \$35, as is seen in the figure below.



Now, suppose we were to buy one unit of the commodity and sell n_c call options. The terminal value of this portfolio is \$45 − $5n_c$ if the commodity price rises and \$35 if the commodity price falls. The uncertainty of the portfolio's terminal value can be completely eliminated by setting n_c such that

$$45 - 5n_c = 35 \quad \text{or} \quad n_c = 2.$$

In other words, if we buy one unit of the commodity and sell two calls, the terminal value of the portfolio is certain to be \$35. This is the concept of a *riskless hedge portfolio.*

Due to the existence of this riskless hedge portfolio, we can price the European call option in the above example. The cost of forming this riskless hedge portfolio at time 0 is \$40 − \$2*c*. Since the investment of \$40 − \$2*c* provides a certain terminal value of \$35, it must be the case that if we would alternatively invest the \$40 − \$2*c* in riskless securities we would also realize a terminal value of \$35. If the riskless rate of interest over the three-month interval is 2 percent, then the absence of costless arbitrage opportunities in the marketplace requires that

$$\$(40 - 2c)(1.02) = \$35.$$

In other words, the price of the European call is \$2.84.

The fact that a riskless hedge may be formed between the option and the underlying commodity has an important implication—the price of the risky call option can be derived without knowing the expected rate of return on the commodity. Even though the probabilities of the commodity price moving up to \$45 or down to \$35 were not known in the above example, we were still able to price the option. In other words, the value of the call relative to the commodity is not influenced by investor preferences. It does not matter whether an individual is risk-averse or risk-neutral, both are willing to pay \$2.84 for the call option in the above example.

Risk-Neutral Valuation Using the Binomial Model

Cox and Ross (1976) carry this argument one step further. They recognize that, since the price of the call is invariant to investor preferences, nothing is lost if we assume that everyone is risk-neutral. Under an assumption of risk-neutrality, we can find the "risk-neutral" probabilities of an upstate or a downstate in the above example. In a risk-neutral world, the expected terminal value of the commodity is simply its current price times one plus the riskless rate of interest. Hence,

$$\$40(1.02) = \$45p + \$35(1-p),$$

or p equals 58 percent. We then use this probability to compute the expected terminal value of the call option, that is,

$$E(c_T) = \$5(.58) + \$0(.42) = \$2.90.$$

Finally, the current value of the call is simply the present value of the expected terminal value. Under the assumption of risk-neutrality, the discount rate is the riskless rate of interest, so the current call price is

$$c = \frac{\$2.90}{1.02} = \$2.84,$$

exactly the result that we obtained using the riskless hedge portfolio. It is important to remember that this approach prices the option relative to the current commodity price, which is assumed to be "correct."

Risk-Averse Valuation Using the Binomial Model

The price of the option computed using the risk-neutral valuation approach is the same as the price computed using an economy where individuals are assumed to be risk-averse. To see this, consider a binomial framework where the commodity price is currently at $40 and has "risk-averse" probabilities, p', of rising to $45 and $1 - p'$ of falling to $35. Suppose that the expected rate of return on the commodity is 4 percent over the next three months, where the riskless rate of interest is 2 percent. The difference between the two rates reflects the risk premium demanded by individuals for holding the risky commodity. If the expected rate of return on the commodity is 4 percent, then the risk-averse probabilities are determined by

$$\$40(1.04) = \$45p' + \$35(1-p'),$$

that is, p' is 66 percent. The higher probability of an upstate reflects the fact that the risk-averse individual demands a greater reward for bearing risk than the risk-neutral individual. The expected option price at expiration is, therefore,

$$E(c_T) = \$5(.66) + \$0(.34) = \$3.30.$$

The next step in valuation is to determine the appropriate risk-adjusted discount rate for the call option. In a risk-neutral economy, the rate is simply the riskless rate of interest since individuals are indifferent about risk. Risk-averse individuals, however, care about risk and demand higher rates of return for risky assets or commodities. For example, under the capital asset pricing model, the expected rate of return on the commodity is

$$E_S = r + (E_M - r)\beta_S,$$

where E_S and E_M are the expected returns for the commodity and the market portfolio, respectively, r is the riskless rate of return, and β_S is the commodity's relative systematic risk coefficient. Substituting in the example values, we find

$$.04 = .02 + (E_M - .02)\beta_S$$

or

$$(E_M - .02)\beta_S = .02.$$

Since β_S represents the percentage change in the commodity price with respect to a percentage change in the market portfolio, we can multiply β_S by the percentage change in the call price with respect to a percentage change in the commodity price to obtain the call option's beta and, hence, expected rate of return. That is,

$$\begin{aligned} E_c &= r + (E_M - r)\beta_c \\ &= r + (E_M - r)\beta_S\left(\frac{dc/c}{dS/S}\right). \end{aligned}$$

But, in the case of our illustration, the percentage change in the option price is

$$\frac{dc/c}{dS/S} = \frac{dc}{dS} \times \frac{S}{c} = \frac{5-0}{45-35} \times \frac{40}{c} = \frac{20}{c}.$$

Thus, substituting for $(E_M - .02)\beta_S$, we find

$$E_c = .02 + .02\left(\frac{20}{c}\right).$$

The present value of the call is, therefore,

$$c = \frac{E(c_t)}{1+E_c} = \frac{3.30}{1+.02+\frac{.40}{c}},$$

so the call price is

$$c = \frac{3.30-.40}{1.02} = 2.84,$$

exactly the same result as obtained for the risk-neutral economy.

11.2 COMMODITY PRICE AND RETURN DISTRIBUTIONS

The valuation of European call options is nearly as simple as the illustration shows. The only additional feature that must be incorporated in the valuation of the European call option is a more realistic assumption about the distribution of the commodity price at the time the option expires. This section deals with the distributional characteristics of the commodity prices and returns.

Before discussing specific distributional properties, a few basic definitions are required. First, consider a sequence of periodic commodity prices beginning today and continuing through time T,[2]

$$S_0, \tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_T.$$

The random *rate of return* on the commodity over the T periods is defined as being the price relative less one, that is,

$$\tilde{S}_T/S_0 - 1. \tag{11.1}$$

The random *continuously compounded rate of return* over the T periods is

$$\tilde{x} = \ln(\tilde{S}_T/S_0), \tag{11.2}$$

or, alternatively, the random *terminal commodity price* is

$$\tilde{S}_T = S_0 e^{\tilde{x}}. \tag{11.3}$$

Note that the continuously compounded T-period return is the sum of the T continuously compounded periodic returns, that is,

[2]These commodity prices are observed at intervals equally spaced through time.

$$\ln(\tilde{S}_T/S_0) = \sum_{t=1}^{T} \ln(\tilde{S}_t/\tilde{S}_{t-1}). \tag{11.4}$$

An assumption that is commonly used in the development of finance models is that security returns are independently and identically distributed each period. Thus, the expected continuously compounded periodic return is

$$E[\ln(\tilde{S}_t/\tilde{S}_{t-1})] = \mu, \tag{11.5}$$

and, by equation (11.4), the expected continuously compounded return from 0 to T is

$$E[\ln(\tilde{S}_T/S_0)] = \sum_{t=1}^{T} E[\ln(\tilde{S}_t/\tilde{S}_{t-1})] = \mu T. \tag{11.6}$$

Similarly, the variance of the continuously compounded periodic return is

$$\text{Var}[\ln(\tilde{S}_t/\tilde{S}_{t-1})] = \sigma^2, \tag{11.7}$$

so the variance of the continuously compounded return from 0 to T is

$$\text{Var}[\ln(\tilde{S}_T/S_0)] = \sum_{t=1}^{T} \text{Var}[\ln(\tilde{S}_t/\tilde{S}_{t-1})] = \sigma^2 T. \tag{11.8}$$

The first and the second terms in (11.8) are equal by virtue of the assumption of independence between returns in different periods. The standard deviation of the continuously compounded return from 0 to T is $\sigma\sqrt{T}$.

The second assumption that we invoke is that the continuously compounded periodic rates of return are normally distributed with mean μ and variance σ^2. In this case, the continuously compounded return from 0 to T is also normally distributed with mean μT and variance $\sigma^2 T$. It also implies that the distribution of stock prices is lognormal with mean

$$E(\tilde{S}_T) = S_0 e^{\alpha T}, \tag{11.9}$$

where

$$\alpha = \mu + \sigma^2/2. \tag{11.9a}$$

(A proof of this is contained in Appendix 11.1.)

Figures 11.3a and 11.3b contain illustrations of the two distributions that we are implicitly using. The first is the normal distribution for $\tilde{x}$, which has mean μT

FIGURE 11.3(a) Normal Distribution

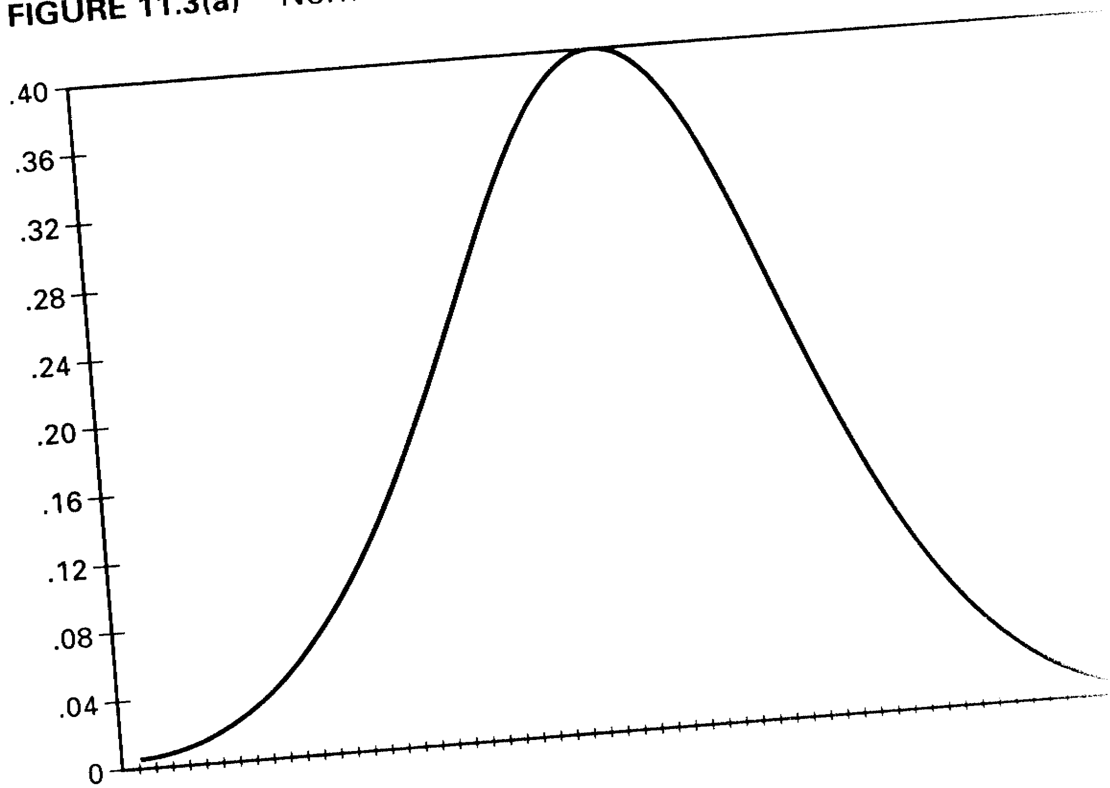
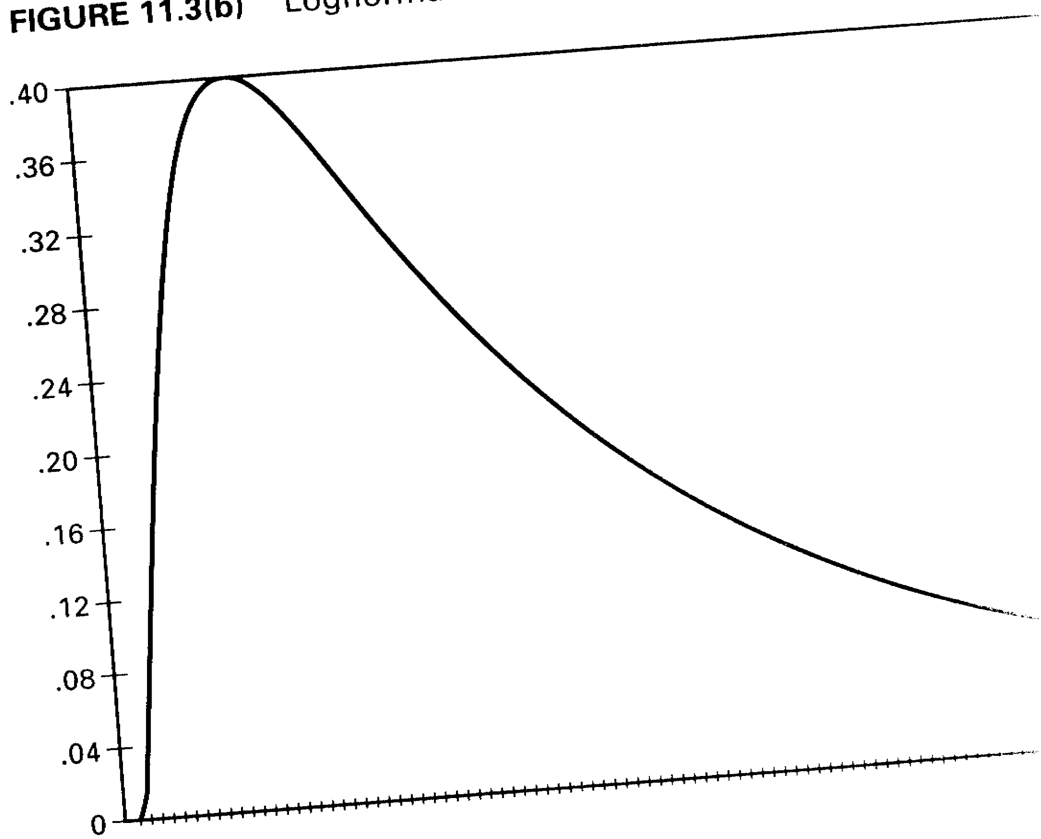


FIGURE 11.3(b) Lognormal Distribution



and variance $\sigma^2 T$. The second is the lognormal distribution of $\tilde{S}_T$, which has mean $S_0 e^{\alpha T}$. Note that the price distribution has the intuitively appealing characteristic that the terminal commodity price cannot fall below zero. If terminal prices were assumed to be normally distributed, there would be some chance that the commodity price would go below zero.

Our use of the normal distribution is further facilitated by transforming the continuously compounded return, $\tilde{x}$ or $\ln(\tilde{S}_T/S_0)$, into a standard normally distributed variable, $\tilde{z}$, which has mean zero and variance one, that is,

$$\tilde{z} = \frac{\tilde{x} - \mu T}{\sigma\sqrt{T}} = \frac{\ln(\tilde{S}_T/S_0) - \mu T}{\sigma\sqrt{T}}, \tag{11.10}$$

which may also be written in terms of the terminal commodity price

$$\tilde{S}_T = S_0 e^{\mu T + \sigma\sqrt{T}\tilde{z}}. \tag{11.11}$$

The variable $\tilde{z}$ has the density function

$$n(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}. \tag{11.12}$$

The probability that a drawing from this unit normal distribution will produce a value less than the constant, d, is

$$\begin{aligned}\text{Prob}(\tilde{z} < d) &= \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ &= N(d).\end{aligned} \tag{11.13}$$

To evaluate the probability $N(d)$ in (11.13), a variety of methods can be used. Polynomial approximations are popular because they are simple to program. Appendix 11.2 contains two such approximations and their levels of accuracy. Another option is to use the values of the normal probabilities tabulated in statistics textbooks and other publications. Appendix 11.3 contains normal probabilities tabulated over the range of d from -4.99 to $+4.99$.

Two properties of the cumulative unit normal density function will prove useful later in this chapter. First, the probability of drawing a value greater than d from a unit normal distribution equals one minus the probability of drawing a value less than d,[3] that is,

$$\text{Prob}(\tilde{z} \geq d) = 1 - N(d). \tag{11.14}$$

[3]This result follows simply from $\text{Prob}(\tilde{z} < d) + \text{Prob}(\tilde{z} \geqslant d) = 1$ and (11.13).

Second, since the unit normal distribution is symmetric around 0, the probability of drawing a value less than d equals one minus the probability of drawing a value less than $-d$,[4] that is,

$$N(d) = 1 - N(-d). \tag{11.15}$$

EXAMPLE 11.1

Compute the probabilities that a drawing from a normal distribution will provide a value that is (a) within one standard deviation of the mean, (b) within two standard deviations of the mean, and (c) within three standard deviations of the mean.

First, it should be noted that any normally distributed variable, $\tilde{x}$, can be transformed into a unit normally distributed variable (i.e., a variable with mean zero and variance one) by applying the transformation (11.10). Second, we assess the probabilities using the tabulated values for the cumulative unit normal distribution. (See Appendix 11.3)

$$\begin{aligned}\text{Prob}(-1 \le \tilde{z} \le 1) &= \text{Prob}(\tilde{z} \le 1) - \text{Prob}(\tilde{z} \le -1)\\ &= .84134 - .15866 = .68268\end{aligned}$$

$$\begin{aligned}\text{Prob}(-2 \le \tilde{z} \le 2) &= \text{Prob}(\tilde{z} \le 2) - \text{Prob}(\tilde{z} \le -2)\\ &= .97725 - .02275 = .95450\end{aligned}$$

$$\begin{aligned}\text{Prob}(-3 \le \tilde{z} \le 3) &= \text{Prob}(\tilde{z} \le 3) - \text{Prob}(\tilde{z} \le -3)\\ &= .99865 - .00135 = .99730\end{aligned}$$

EXAMPLE 11.2

Assume that the current commodity price is \$50 and that the continuously compounded rate of return has an annualized mean of 16 percent and a standard devi-

[4] This result is derived as follows:

$$\begin{aligned}N(d) &= \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz,\\ &= \int_{-d}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz\\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz - \int_{-\infty}^{-d} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz,\\ &= 1 - N(-d).\end{aligned}$$

ation of 32 percent. Compute the probability that the commodity price will exceed 75 at the end of three months.

First, we use equation (11.10) to transform the lognormal terminal price to a unit normal variable value. Specifically,

$$d = \frac{\ln(75/50) - .16(.25)}{.32\sqrt{.25}} = 2.28416.$$

Second, we round d to the nearest one-hundredth and use the probability tables:

$$\text{Prob}(\tilde{S}_T < 75) = \text{Prob}(\tilde{z} < 2.28) = .98870.$$

Note that we are evaluating the probability of the terminal commodity price being less than 75 because the tables find the area under the unit normal density function from minus infinity up to the limit d. To compute the probability that the terminal commodity price will be greater than 75, we simply take the complement or

$$\begin{aligned}\text{Prob}(\tilde{S}_T \geq 75) &= 1 - \text{Prob}(\tilde{S}_T < 75)\\ &= 1 - \text{Prob}(\tilde{z} < 2.28)\\ &= 1 - .98870 = .01130.\end{aligned}$$

Note that we are introducing some error as a result of rounding the upper integral limit d to the nearest hundredth when using the tables. We could interpolate between table values to achieve greater accuracy, or we could use one of the polynomial approximations in Appendix 11.2. Using the second polynomial approximation in the appendix provides

$$N(2.28416) = .98882.$$

EXAMPLE 11.3

Using the parameters from Example 11.2, compute the probability that the commodity price will fall between 40 and 60 at the end of six months.

Again, we use equation (11.10) to transform the terminal commodity price to a unit normally distributed variable. The limits of integration are

$$d_1 = \frac{\ln(60/50) - .16(.5)}{.32\sqrt{.5}} = .45220$$
$$d_2 = \frac{\ln(40/50) - .16(.5)}{.32\sqrt{.5}} = -1.33972$$

The probabilities are

$$\text{Prob}(\tilde{z} < .45220) = N(.45220) = .67444$$
$$\text{Prob}(\tilde{z} < -1.33972) = N(-1.33972) = .09017,$$

where the probabilities were again computed using the second polynomial approximation in Appendix 11.2.

The final step involves differencing the probabilities, that is,

$$\begin{aligned}\text{Prob}(40 \leq \tilde{S}_T \leq 60) &= \text{Prob}(-1.33972 \leq \tilde{z} \leq .45220)\\ &= \text{Prob}(\tilde{z} \leq .45220) - \text{Prob}(\tilde{z} \leq -1.33972)\\ &= .67444 - .09017 = .58427.\end{aligned}$$

EXAMPLE 11.4

Using the parameters from Example 11.2, compute the range of the commodity price in three months assuming that it will be within two standard deviations of its current level.

Use equation (11.11) and set $\tilde{z}$ equal to ± 2. The two terminal commodity prices are

$$S_{T1} = 50e^{.16(.25)+.32\sqrt{.25}(-2)} = 37.78919$$
$$S_{T2} = 50e^{.16(.25)+.32\sqrt{.25}(2)} = 71.66647.$$

EXAMPLE 11.5

Suppose that there is a three-month European call option written on the commodity in Example 11.2 and its exercise price is 50. Compute the probability that the call option will be in-the-money at expiration. The upper integral limit d is

$$d = \frac{\ln(50/50) - .16(.25)}{.32\sqrt{.25}} = -.25000.$$

The probability that the terminal commodity price will be less than the exercise price is

$$\text{Prob}(\tilde{S}_T < 50) = N(-.25000) = .40129,$$

so the probability that the commodity price will exceed the exercise price is

$$\text{Prob}(\tilde{S}_T \geq 50) = 1 - N(-.25000) = .59871.$$

EXAMPLE 11.6

Compute the expected rate of return on the commodity over a three-month interval and the expected commodity price at that point in time.

By equation (11.9a), we know that the expected rate of return on the commodity is equal to the mean plus one-half of the variance of the distribution of the logarithm of the commodity price ratio, $\ln(\tilde{S}_T/S)$, that is,

$$\begin{aligned}\alpha &= \mu + \sigma^2/2 \\ &= .16 + .32^2/2 \\ &= .2112.\end{aligned}$$

The expected terminal commodity price is, therefore,

$$E(\tilde{S}_T) = Se^{\alpha T} = 50e^{.2112(.25)} = 52.71094.$$

11.3 RISK-NEUTRAL VALUATION OF EUROPEAN CALL OPTION

The European call option valuation equation is now derived under the distributional assumptions discussed in the previous section. The valuation approach is consistent with the numerical illustration used in Section 11.1—first, we estimate the expected terminal value of the call, and then we discount the expected terminal value to the present. The theoretical call price is simply

$$c = e^{-rT}E(\tilde{c}_T). \tag{11.16}$$

To evaluate the expected terminal call price, we assume that the expected rate of return on the commodity equals the riskless rate of interest (risk-neutrality) and that the commodity prices are lognormally distributed at the option's expiration. To discount the expected terminal call price to the present, we assume that the expected rate of return on the call equals the riskless rate of interest (risk-neutrality).

In order to make equation (11.16) operational, we need to evaluate the term $E(\tilde{c}_T)$, the expected terminal value of the call option. If $\tilde{S}_T$ is assumed to be lognormally distributed, the distribution of the terminal call price, $\tilde{c}_T$, is known since $\tilde{c}_T$ is simply

$$\tilde{c}_T = \begin{cases} \tilde{S}_T - X & \text{for } S_T \geq X \\ 0 & \text{for } S_T < X. \end{cases} \tag{11.17}$$

With the terminal commodity price having a lognormal distribution, condition (11.17) implies that the terminal call price has a truncated lognormal distribution and that the mean of the call price distribution is:

$$\begin{aligned} E(\tilde{c}_T) &= E(\tilde{S}_T - X|S_T \geq X) + E(0|S_T < X) \\ &= E(\tilde{S}_T - X|S_T \geq X) \\ &= E(\tilde{S}_T|S_T \geq X) - E(X|S_T \geq X) \\ &= E(\tilde{S}_T|S_T \geq X) - X\text{Prob}(S_T \geq X), \end{aligned} \tag{11.18}$$

where $\text{Prob}(S_T \geqslant X)$ is the probability that the commodity price exceeds the option's exercise price at expiration. To evaluate $E(\tilde{c}_T)$, we must evaluate each of the two terms on the right-hand side of (11.18). We will begin with the second term.

Evaluation of $X\text{Prob}(S_T \geqslant X)$

Letting $L(S_T)$ be the lognormal density function of S_T, the term $X\text{Prob}(S_T \geqslant X)$, is

$$X\text{Prob}(S_T \geq X) = X \int_X^{+\infty} L(S_T)dS_T.$$

The easiest way to evaluate the integral is to perform a change of variables on S_T. Equation (11.10) shows us the transformation that we apply to S_T. The upper and lower limits of integration for the new variable z are obtained by substituting $+\infty$ and X for S_T in (11.10). The limits are therefore $+\infty$ and $[\ln(X/S) - \mu T]/\sigma\sqrt{T}$, respectively. Thus,

$$\begin{aligned} X\text{Prob}(S_T \geq X) &= X \int_{\frac{\ln(X/S)-\mu T}{\sigma\sqrt{T}}}^{+\infty} n(z)dz \\ &= X \int_{-\infty}^{\frac{\ln(S/X)+\mu T}{\sigma\sqrt{T}}} n(z)dz \\ &= XN(d_2), \end{aligned} \tag{11.19}$$

where $d_2 = [\ln(S/X) + \mu T]/(\sigma\sqrt{T})$. In other words, the value $N(d_2)$ is the probability that the commodity price will exceed the exercise price at the option's expiration.

Evaluation of $E(\tilde{S}_T|S_T \geqslant X)$

The evaluation of the first term of equation (11.18), $E(\tilde{S}_T|S_T \geqslant X)$, is slightly more difficult. The initial steps are as follows:

$$
\begin{aligned}
E(\tilde{S}_T|S_T \geq X) &= \int_X^{+\infty} S_T L(S_T) dS_T \\
&= \int_{\frac{\ln(X/S)-\mu T}{\sigma\sqrt{T}}}^{+\infty} S e^{\mu T+\sigma\sqrt{T}z} e^{-z^2/2}\frac{1}{\sqrt{2\pi}}dz \\
&= Se^{\mu T}\int_{\frac{\ln(X/S)-\mu T}{\sigma\sqrt{T}}}^{+\infty} e^{\sigma^2 T/2-\sigma^2 T/2+\sigma\sqrt{T}z-z^2/2}\frac{1}{\sqrt{2\pi}}dz \\
&= Se^{\mu T+\sigma^2 T/2}\int_{\frac{\ln(X/S)-\mu T}{\sigma\sqrt{T}}}^{+\infty} e^{-(\sigma\sqrt{T}-z)^2/2}\frac{1}{\sqrt{2\pi}}dz \\
&= Se^{\mu T+\sigma^2 T/2}\int_{-\infty}^{\frac{\ln(S/X)+\mu T}{\sigma\sqrt{T}}+\sigma\sqrt{T}} e^{-y^2/2}\frac{1}{\sqrt{2\pi}}dy \\
&= Se^{\mu T+\sigma^2 T/2}N(d_1),
\end{aligned}
\tag{11.20}
$$

where $d_1 = [\ln(S/X) + \mu T]/(\sigma\sqrt{T}) + \sigma\sqrt{T}$. The steps in (11.20) are as follows: (a) the conditional expected value is expressed in integral form where $L(S_T)$ is the lognormal density function for S_T; (b) a change of variables is performed on S_T, and the density function of the standardized normal variable, z, is written out; (c) $Se^{\mu T}$ is factored out of the integral and the square in the exponent within the integral is completed; (d) $e^{\sigma^2 m2}$ is factored out of the intergral and the remaining expression in the exponent within the integral is simplified; (e) a change of variables $y = \sigma\sqrt{T} - z$ is performed and the limits of the integration are redefined;[5] and (f) the expression is simplified.

Evaluation of $E(\tilde{c}_T)$

To summarize, under the assumption that commodity prices are lognormally distributed and that individuals are risk-neutral, we are attempting to value a European call option. We are in the process of valuing the expected terminal value of the call option, $E(\tilde{c}_T)$. Substituting equations (11.20) and (11.19) into equation (11.18), we now have

$$E(\tilde{c}_T) = Se^{\mu T+\sigma^2 T}N(d_1) - XN(d_2) \tag{11.21}$$

---

[5]Where $y = -z$, the following property holds:

$$\int_{-d}^{\infty} e^{-z^2/2}\frac{1}{\sqrt{2\pi}}dz = \int_{-\infty}^{d} e^{-y^2/2}\frac{1}{\sqrt{2\pi}}dy.$$

This property is used in simplifying (11.20).

where

$$d_1 = \frac{\ln(S/X) + \mu T}{\sigma\sqrt{T}} + \sigma\sqrt{T}, \tag{11.21a}$$

$$d_2 = \frac{\ln(S/X) + \mu T}{\sigma\sqrt{T}}. \tag{11.21b}$$

We will not stop here, however. The expected rate of return of the commodity in the integral limits d_1 and d_2 is the mean of the logarithm of the commodity price ratios—$E[\ln(\tilde{S}_t/\tilde{S}_{t-1})]$. We would like to express the expected rate of return of the commodity in terms of the raw price relatives—$E(\tilde{S}_T/S)$. We know that

$$E(\tilde{S}_T/S) = e^{\alpha T} = e^{(\mu+\sigma^2/2)T}. \tag{11.22}$$

Now, recall that we have invoked an assumption of risk-neutrality. The value of α in (11.22) is the expected rate of return on the commodity, and, in a risk-neutral world, the expected rate of return on the commodity equals the cost-of-carry rate, b (i.e., the cost of interest plus any additional costs). Substituting b for α in (11.9a) and isolating μ, we get

$$\mu = b - \sigma^2/2. \tag{11.23}$$

Substituting this into (11.21), we get

$$E(\tilde{c}_T) = Se^{bT}N(d_1) - XN(d_2), \tag{11.24}$$

where

$$d_1 = \frac{\ln(S/X) + (b + .5\sigma^2)T}{\sigma\sqrt{T}}, \tag{11.24a}$$

$$\begin{aligned} d_2 &= \frac{\ln(S/X) + (b - .5\sigma^2)T}{\sigma\sqrt{T}}, \\ &= d_1 - \sigma\sqrt{T}. \end{aligned} \tag{11.24b}$$

Current Value of Call

With an explicit valuation of $E(\tilde{c}_T)$ in hand, we can substitute into equation (11.16) to find the *valuation formula for the European call option on a commodity with cost-of-carry rate b,* that is,

$$c(S, T; X) = Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2), \tag{11.25}$$

where

$$d_1 = \frac{\ln(S/X) + (b + .5\sigma^2)T}{\sigma\sqrt{T}}, \quad \textbf{(11.25a)}$$

$$d_2 = d_1 - \sigma\sqrt{T}. \quad \textbf{(11.25b)}$$

The interpretation of the terms in (11.25) is fairly straightforward given our risk-neutral valuation approach. The term $Se^{(b-r)T}N(d_1)$ is the present value of the expected benefit of exercising the call option at expiration, conditional on the terminal commodity price being greater than the exercise price at the option's expiration. The term $N(d_2)$ is the probability that the commodity price will be greater than the exercise price at expiration. The present value of the expected cost of exercising the call option conditional upon the call being in-the-money at expiration is $Xe^{-rT}N(d_2)$.

EXAMPLE 11.7

Compute the price of a three-month European foreign currency call option with an exercise price of 40. The spot exchange rate is currently 40, the domestic interest rate is 8 percent annually, the foreign interest rate is 12 percent annually, and the standard deviation of the continuously compounded return is 30 percent on an annualized basis. Note that the cost-of-carry rate, b, is, therefore, -4 percent.

$$c = 40e^{(-.04-.08).25}N(d_1) - 40e^{-.08(.25)}N(d_2),$$

where

$$d_1 = \frac{\ln(40/40) + [-.04 + .5(.30)^2](.25)}{.30\sqrt{.25}} = .0083,$$

$$d_2 = d_1 - .30\sqrt{.25} = -.1417.$$

The values of $N(d_1)$ and $N(d_2)$ are .5033 and .4437, respectively, so the European call option price is

$$c = 38.818(.5033) - 39.208(.4437) = 2.14.$$

11.4 RISK-NEUTRAL VALUATION OF EUROPEAN PUT OPTION

The risk-neutral valuation approach can be applied to the European put option pricing problem to find the put's valuation equation. A simpler way, however, is to combine the European put-call parity relation from the last chapter with the European call option valuation equation (11.25). In the absence of costless arbitrage opportunities in the marketplace, the European put-call parity relation,

$$c(S,T;X) - p(S,T;X) = Se^{(b-r)T} - Xe^{-rT}, \qquad \textbf{(11.26)}$$

holds at all points in time. Isolating $p(S,T;X)$ in (11.26), we get

$$p(S,T;X) = c(S,T;X) - Se^{(b-r)T} + Xe^{-rT}. \qquad \textbf{(11.27)}$$

Substituting the European call option valuation equation (11.25) for the term $c(S,T;X)$, we find that the *valuation equation for a European put option on a commodity* is

$$\begin{aligned} p(S,T;X) &= Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2) - Se^{(b-r)T} + Xe^{-rT} \\ &= Xe^{-rT}[1 - N(d_2)] - Se^{(b-r)T}[1 - N(d_1)] \\ &= Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1), \end{aligned} \qquad \textbf{(11.28)}$$

where

$$d_1 = \frac{\ln(S/X) + (b + .5\sigma^2)T}{\sigma\sqrt{T}}, \qquad \textbf{(11.28a)}$$

$$d_2 = d_1 - \sigma\sqrt{T}. \qquad \textbf{(11.28b)}$$

Thus, the valuation of the European put option follows straightforwardly from European put-call parity and the valuation of the European call option.

The interpretation of the terms in (11.28) parallels the risk-neutral interpretation for the call. The term $Xe^{-rT}N(-d_2)$ is the present value of the expected benefit of exercising the put option at expiration conditional upon the terminal commodity price being less than the exercise price at the option's expiration. Recall the put option provides the right to sell the commodity so the benefit from holding the option is the cash we receive when we exercise the option, that is, X. $N(-d_2)$ is the probability that the commodity price will be less than the exercise price at expiration. Note that it is the complement of $N(d_2)$, the probability that the terminal commodity price will exceed the exercise price. The present value of the expected cost of exercising the put option conditional upon the put option being in-the-money at expiration is $Se^{(b-r)T}N(-d_1)$. If we exercise the put, we must forfeit the commodity as fulfillment of our obligation so the present value of the expected terminal commodity price conditional upon exercise is our cost today.

EXAMPLE 11.8

Compute the price of a three-month European foreign currency put option with an exercise price of 40. The spot exchange rate is currently 40, the domestic interest rate is 8 percent annually, the foreign interest rate is 12 percent annually, and the standard deviation of the continuously compounded return is 30 percent on an annualized basis. Note that the cost-of-carry rate, b, is, therefore, -4 percent.

$$p = 40e^{-.08(.25)}N(-d_2) - 40e^{(-.04-.08).25}N(-d_1),$$

where

$$d_1 = \frac{\ln(40/40) + [-.04 + .5(.30)^2](.25)}{.30\sqrt{.25}} = .0083,$$

$$d_2 = d_1 - .30\sqrt{.25} = -.1417.$$

The values of $N(-d_2)$ and $N(-d_1)$ are .5563 and .4967, respectively, so the European put option price is

$$p = 39.208(.5563) - 38.818(.4967) = 2.53.$$

Note that this result, together with the result of Exercise 11.7 verifies the put-call parity relation (11.26), that is,

$$2.14 - 2.53 = 38.818 - 39.208.$$

11.5 PROPERTIES OF THE EUROPEAN CALL AND PUT OPTION PRICING FORMULAS

The valuation equations for the European call and put options are

$$c(S,T;X) = Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2) \tag{11.25}$$

and

$$p(S,T;X) = Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1), \tag{11.28}$$

respectively, where

$$d_1 = \frac{\ln(S/X) + (b + .5\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

The option price depends on six variables—S, X, b, r, σ, T. In this section, we analyze how the European call and put option prices respond to changes in the underlying option variables. Each of the variables will be discussed in turn beginning with the commodity price. The derivations of each of the expressions below are contained in Appendix 11.4.

Change in Commodity Price

The change in the option price with respect to a change in the commodity price is called the option's *delta*. The delta of a European call option is

$$\Delta_c = \frac{\partial c}{\partial S} = e^{(b-r)T} N(d_1) > 0. \tag{11.29a}$$

The call option's delta is unambiguously positive in sign, implying that an increase in commodity price causes the call price to increase. The result is intuitive since the call option conveys the right to buy the underlying commodity at a fixed price and the underlying commodity has just become more valuable.

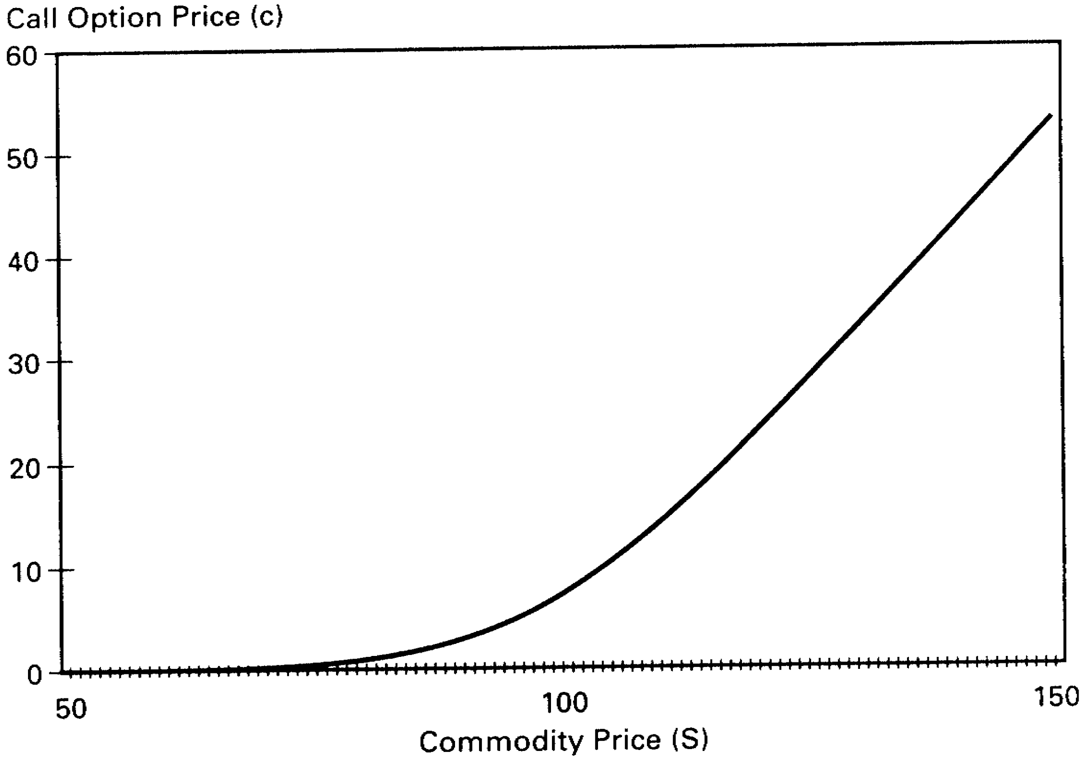
Figure 11.4 shows the how the value of a European call option changes as the underlying commodity price changes. The option has three months to expiration. Notice that when the call option is out-of-the-money, its slope is fairly flat. Out-of-the-money call options have very low delta values; that is, they do not respond very quickly to changes in the commodity price. As the commodity price increases and the call becomes at-the-money and then in-the-money, the slope becomes steeper and steeper. Where the option is very deep in-the-money, the delta value is nearly one, and the call price changes in a one-to-one correspondence with the commodity price. Figure 11.5 shows the option's delta value as a function of the commodity price.

The put option's delta is

$$\Delta_p = \frac{\partial p}{\partial S} = -e^{(b-r)T} N(-d_1) < 0. \tag{11.29b}$$

This derivative is negative because an increase in the commodity price makes the put option less valuable. Again, it can be shown that the sensitivity of the put price

FIGURE 11.4 European call option price (c) as a function of the underlying commodity price (S). The commodity price range is from 50 through 150. The option has an exercise price (X) of 100 and a time to expiration (T) of three months. The cost-of-carry rate (b) is 8 percent, and the riskless rate of interest (r) is 8 percent. The standard deviation of the logarithm of the commodity price ratios (σ) is 30 percent.

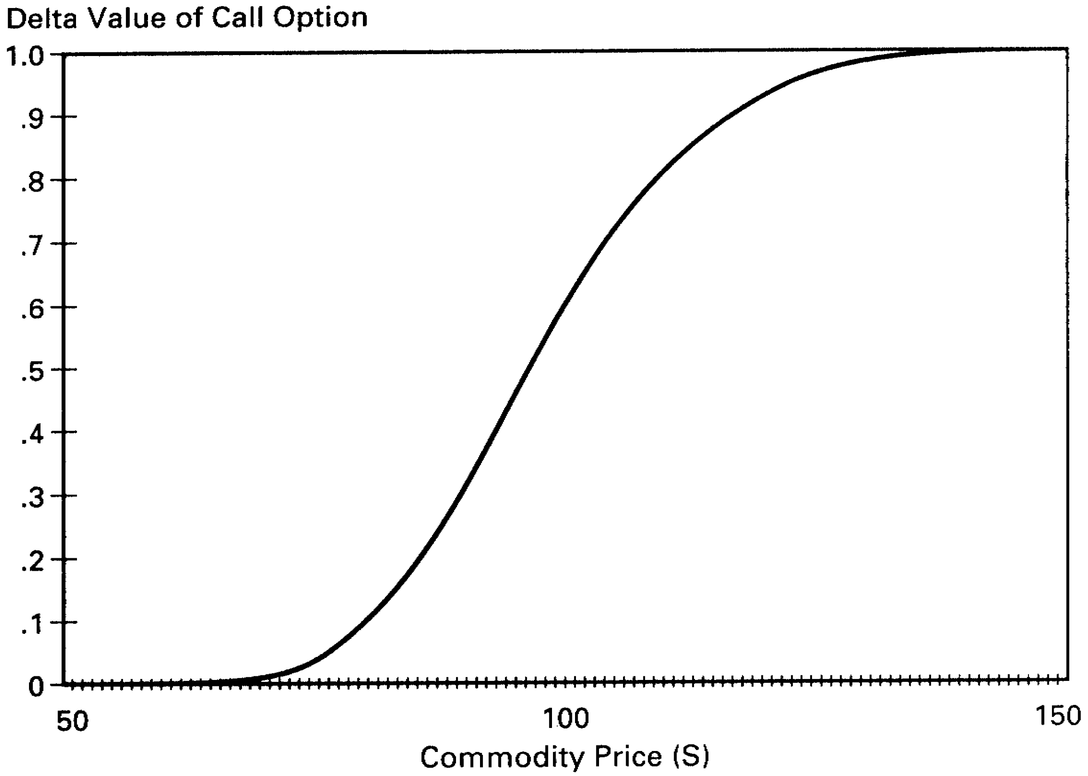


to changes in the underlying commodity price is itself sensitive to the "moneyness" of the option. Figure 11.6 shows this sensitivity for a European put option with three months to expiration. The in-the-money option has a steeper slope than the at-the-money option, which, in turn, has a steeper slope than the out-of-the-money option. Figure 11.7 shows the put's delta value as a function of the underlying commodity price.

Percentage Change in the Commodity Price

It is often the case that, instead of the dollar change in the option price with respect to a dollar change in the underlying commodity price, one is interested in the elasticity of the option price with respect to the commodity price. This elasticity, called the option's *eta*, is the percentage change in the option price with respect to the

FIGURE 11.5 European call option delta (Δ_c) as a function of the underlying commodity price (S). The commodity price range is from 50 through 150. The option has an exercise price (X) of 100 and a time to expiration (T) of three months. The cost-of-carry rate (b) is 8 percent, and the riskless rate of interest (r) is 8 percent. The standard deviation of the logarithm of the commodity price ratios (σ) is 30 percent.



percentage change in the commodity price. The elasticity of the call price with respect to the commodity price is greater than one,[6] that is,

$$\eta_c = \Delta_c \frac{S}{c} = e^{(b-r)T} N(d_1) \frac{S}{c} > 1, \tag{11.30a}$$

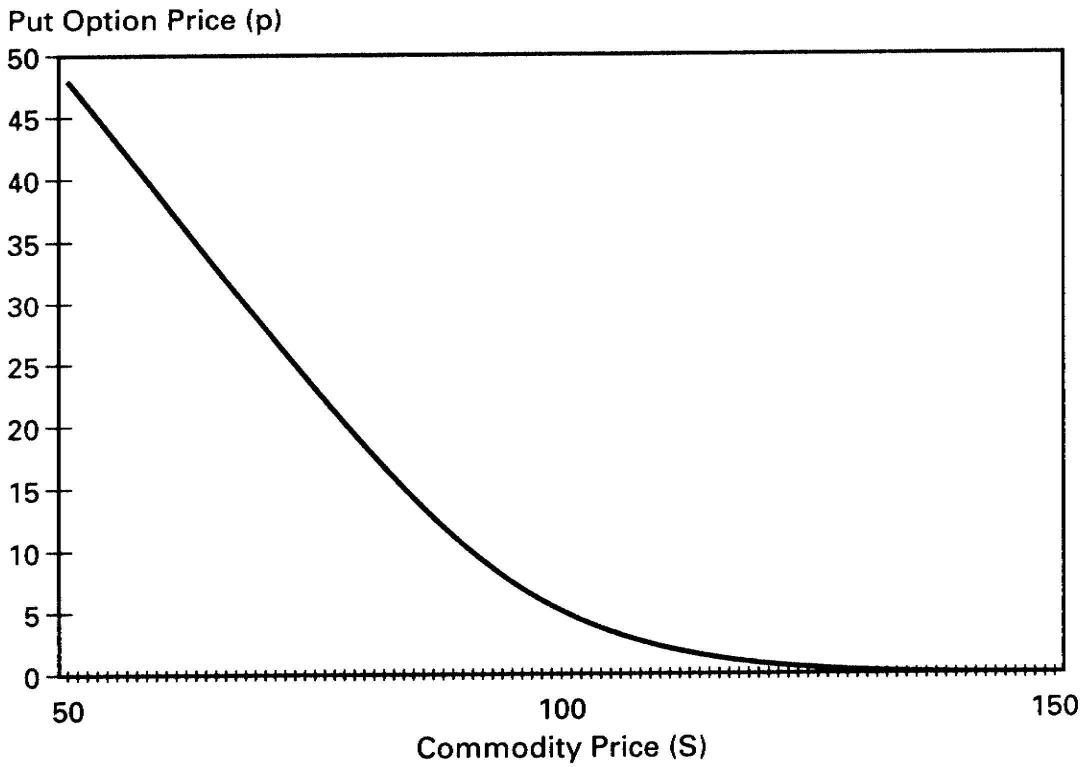
---

[6]The elasticity of the call price with respect to the commodity price can be shown to be greater than one by rewriting (11.30a) as

$$\eta_c = \frac{1}{1 - \frac{Xe^{-rT}N(d_2)}{Se^{(b-r)T}N(d_1)}}.$$

The last term in the denominator is less than one because the European call price cannot be less than zero, therefore, the value of η_c must be greater than one.

FIGURE 11.6 European put option price (p) as a function of the underlying commodity price (S). The commodity price range is from 50 through 150. The option has an exercise price (X) of 100 and a time to expiration (T) of three months. The cost-of-carry rate (b) is 8 percent, and the riskless rate of interest (r) is 8 percent. The standard deviation of the logarithm of the commodity price ratios (σ) is 30 percent.



and the elasticity of the put price with respect to the commodity price is less than minus one,[7] that is,

$$\eta_p = \Delta_p \frac{S}{p} = -e^{(b-r)T} N(-d_1) \frac{S}{p} < -1. \qquad \textbf{(11.30b)}$$

Table 11.1 contains option prices, delta values and elasticities for alternative prices of the underlying commodity. It is interesting to note that (i) the elasticities have very large magnitudes and (ii) the magnitudes are larger for farther out-of-the-money options. If someone has a strong belief that the price of an individual commodity will rise, an investment in a call option will provide a larger rate of

[7]The proof that η_p is less than -1 follows along the same lines as the proof that $\eta_c > 1$ in the previous footnote.

FIGURE 11.7 European put option delta (Δ_p) as a function of the underlying commodity price (S). The commodity price range is from 50 through 150. The option has an exercise price (X) of 100 and a time to expiration (T) of three months. The cost-of-carry rate (b) is 8 percent, and the riskless rate of interest (r) is 8 percent. The standard deviation of the logarithm of the commodity price ratios (σ) is 30 percent.

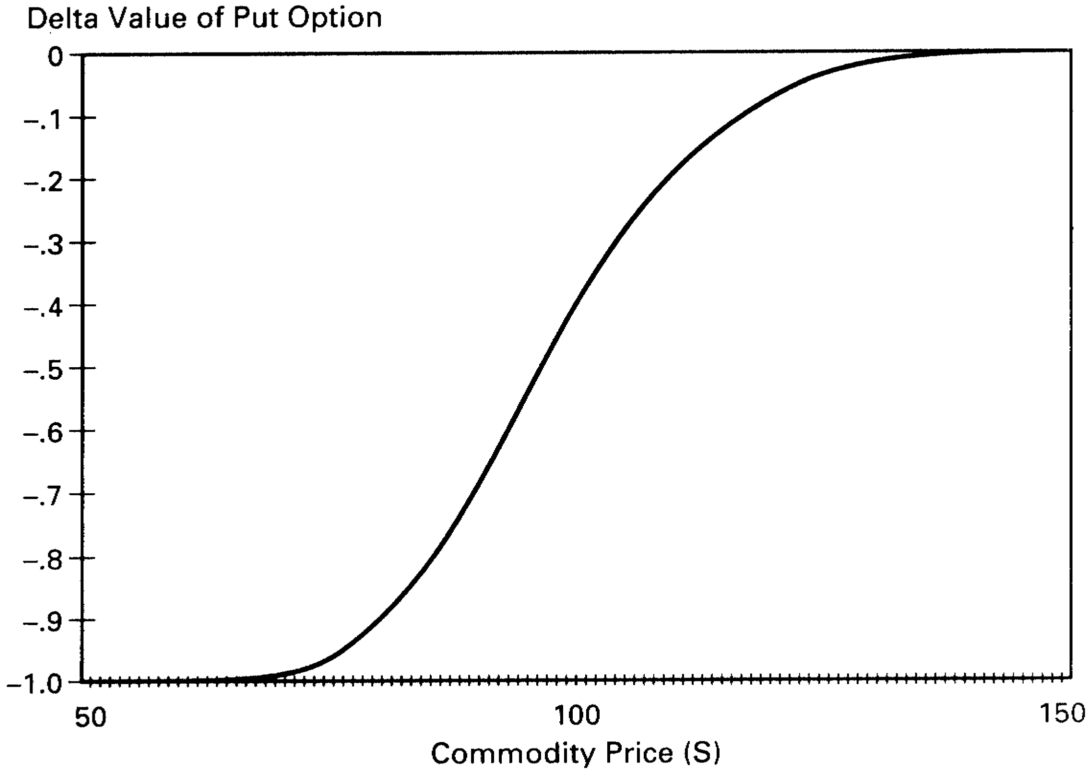


TABLE 11.1 Simulated stock index call and put option prices, deltas, etas, and gammas for option parameters: $X = 100$, $b = .08$, $r = .08$, $T = .25$, and $\sigma = .30$.

Commodity Price S	Call Price c	Delta Δ_c	Eta η_c	Gamma γ_c	Put Price p	Delta Δ_p	Eta η_p	Gamma γ_p
80	.537	.100	14.952	.014	18.557	–.899	–3.878	.014
90	2.494	.310	11.207	.026	10.514	–.689	–5.900	.026
100	6.961	.582	8.367	.026	4.981	–.417	–8.380	.026
110	13.954	.800	6.310	.016	1.974	–.199	–11.109	.016
120	22.645	.922	4.889	.008	.665	–.077	–13.924	.008

return than an investment directly in the commodity and, moreover, an investment in an out-of-the-money call will provide a greater rate of return than an in-the-money call.

These greater rates of return are not without a corresponding increase in risk, however. In fact, just as the rates of return on the options are proportionally related to the rate of return on the commodity, the risk or "beta" of an option is proportionally related to the beta of the commodity, that is, $\beta_c = \eta_c\beta_S$ and $\beta_p = \eta_p\beta_S$. The increase in the expected rate of return as a result of holding a call option is exactly what is justified on the basis of the capital asset pricing model.

Change in Delta

Earlier we described the option's delta, how the option price changes as the commodity price changes. Related to this concept is the option's *gamma*—the change in delta as the commodity price changes. The expression for the gamma of a call option is

$$\gamma_c = \frac{\partial \Delta_c}{\partial S} = \frac{e^{(b-r)T}n(d_1)}{S\sigma\sqrt{T}} > 0, \tag{11.31a}$$

and the gamma for a put is

$$\gamma_p = \frac{\partial \Delta_p}{\partial S} = \frac{e^{(b-r)T}n(d_1)}{S\sigma\sqrt{T}} = \gamma_c > 0, \tag{11.31b}$$

where $n(d_1)$ is the density at d_1. In short, this value tells you how quickly the delta changes as the commodity price changes. Because an option's gamma is largest when the options are approximately at-the-money, these options are the hardest to hedge. In addition, if you believe that the commodity price is about to move in one direction or another (recall the motivation for placing a volatility spread), the at-the-money spread will maximize the portfolio's dollar response to underlying commodity price movements. Figure 11.8 shows the option gamma as a function of the underlying commodity price.

Change in the Exercise Price

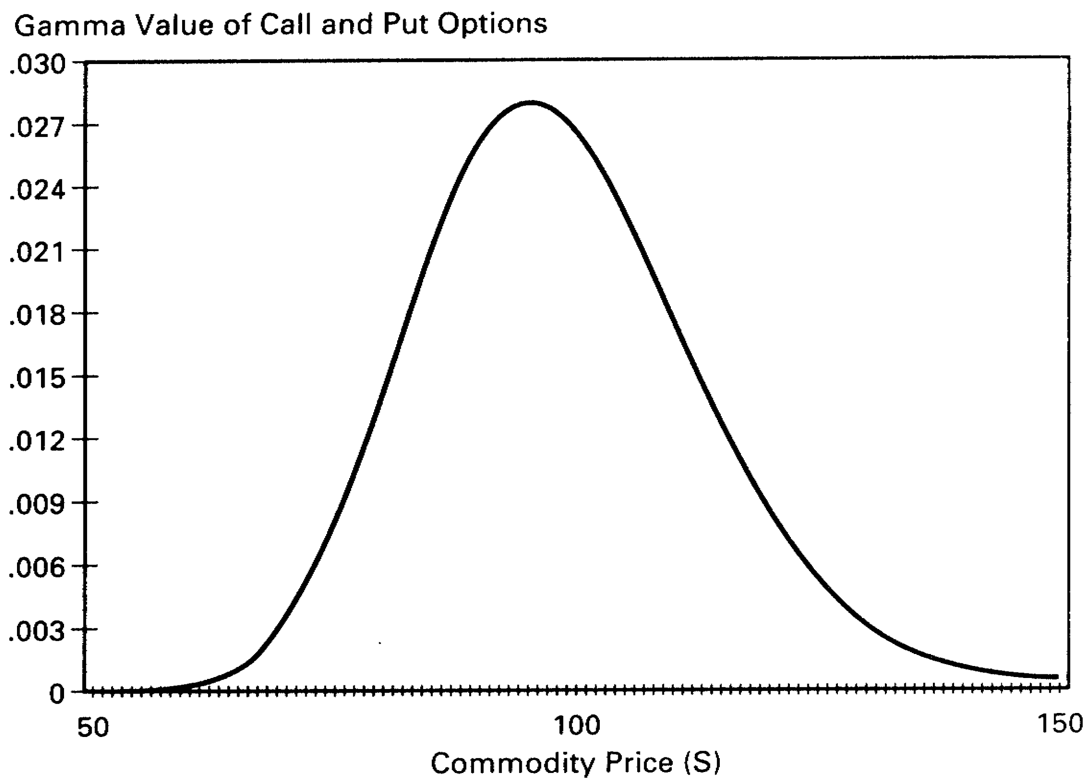
The partial derivatives of the call and put option prices with respect to the exercise price of the option are

$$\frac{\partial c}{\partial X} = -e^{-rT}N(d_2) < 0 \tag{11.32a}$$

and

$$\frac{\partial p}{\partial X} = e^{-rT}N(-d_2) > 0, \tag{11.32b}$$

FIGURE 11.8 European option gamma ($\gamma_c = \gamma_p$) as a function of the underlying commodity price (S). The commodity price range is from 50 through 150. The options have exercise price (X) of 100 and time to expiration (T) of three months. The cost-of-carry rate (b) is 8 percent, and the riskless rate of interest (r) is 8 percent. The standard deviation of the logarithm of the commodity price ratios (σ) is 30 percent.



respectively. Note that, if the exercise price of the options increases, the value of the call option diminishes and the value of the put option increases. This follows from the fact that the call would become more out-of-the-money and the put more in-the-money.

The partial derivatives of the option prices, with respect to the exercise price, are of little practical value in the sense that once the option is created, the exercise price does not change. They are expressed here only in the interest of completeness.

Change in the Cost-of-Carry Rate

The change in the call option price with respect to a change in the cost-of-carry rate is

$$\frac{\partial c}{\partial b} = TSe^{(b-r)T}N(d_1) > 0. \quad \text{(11.33a)}$$

As the cost-of-carry rate increases, the call option value increases, holding constant the spot price and the other variables. The higher the cost of carrying the underlying commodity, the greater the rate of appreciation in the commodity price and hence the greater the call option value. The magnitude of the derivative is small, however. For the foreign currency call option valued in Example 11.7, the partial derivative with respect to the cost-of-carry rate is 4.884. In other words, if the cost-of-carry rate on the underlying commodity increases by 100 basis points, the call price will increase by approximately five cents.

The partial derivative of the put option price with respect to the cost-of-carry rate is

$$\frac{\partial p}{\partial b} = -TSe^{(b-r)T}N(-d_1) < 0. \tag{11.33b}$$

As the cost-of-carry rate increases, the expected rate of appreciation in the commodity price increases and hence the value of the put option declines. The numerical value of this partial derivative for the put option in Example 11.8 is -4.8200.

Change in the Interest Rate

The change in the call option price with respect to a change in the riskless rate of interest is

$$\frac{\partial c}{\partial r} = TXe^{-rT}N(d_2) > 0. \tag{11.34a}$$

The call price increases with an increase in the interest rate because the present value of the exercise price decreases. The value of this derivative is 4.3489 for the call option in Example 11.7.

The partial derivative of the put option price with respect to the riskless rate of interest is

$$\frac{\partial p}{\partial r} = -TXe^{-rT}N(-d_2) < 0. \tag{11.34b}$$

Here the sign is negative because, as the riskless rate of interest increases, the present value of the exercise price received upon exercising the option decreases. The value of the derivative for the put option in Example 11.8 is -5.4531, implying that an increase in the interest rate of 100 basis points reduces the option value by about five cents.

Change in the Volatility

The change in the option price with respect to a change in the volatility[8] of the underlying commodity return is called the option's *vega*. The vega of a European call option is

[8]Up to this point, we have used the term "standard deviation" to describe the dispersion of commodity returns, σ. In the industry, σ is more typically referred to as the *volatility* or the *volatility rate* of the underlying commodity returns, and we adopt that convention for the remainder of the chapter.

$$\text{Vega}_c = \frac{\partial c}{\partial \sigma} = Se^{(b-r)T}n(d_1)\sqrt{T} > 0. \tag{11.35a}$$

The sign of the derivative is positive, indicating that as the volatility of the underlying commodity return increases, the call option value increases. The intuition for this result is that an increase in the volatility rate increases the probability of large upward movements in the underlying commodity price. The probability of large downward commodity price movements also increases, however, it is of no consequence since the call option holder has limited liability.

The numerical value of the call option vega implies that the options price is more sensitive to volatility than it is to either the cost-of-carry rate or the interest rate. The option in Example 11.7 has a vega of 7.7428. An increase in volatility of 100 basis points increases the call's price by nearly eight cents.

The put option's vega is the same as that of the call, that is,

$$\text{Vega}_p = \frac{\partial p}{\partial \sigma} = Se^{(b-r)T}n(d_1)\sqrt{T} = \text{Vega}_c > 0. \tag{11.35b}$$

The put option value also increases with an increase in volatility since the probability increases of a large commodity price decrease. The numerical value of the vega for the put option in Example 11.8 is, therefore, also 7.7428.

Figure 11.9 shows the option's vega as a function of the commodity price. Note that the vega has its highest value where the option is approximately at-the-money.

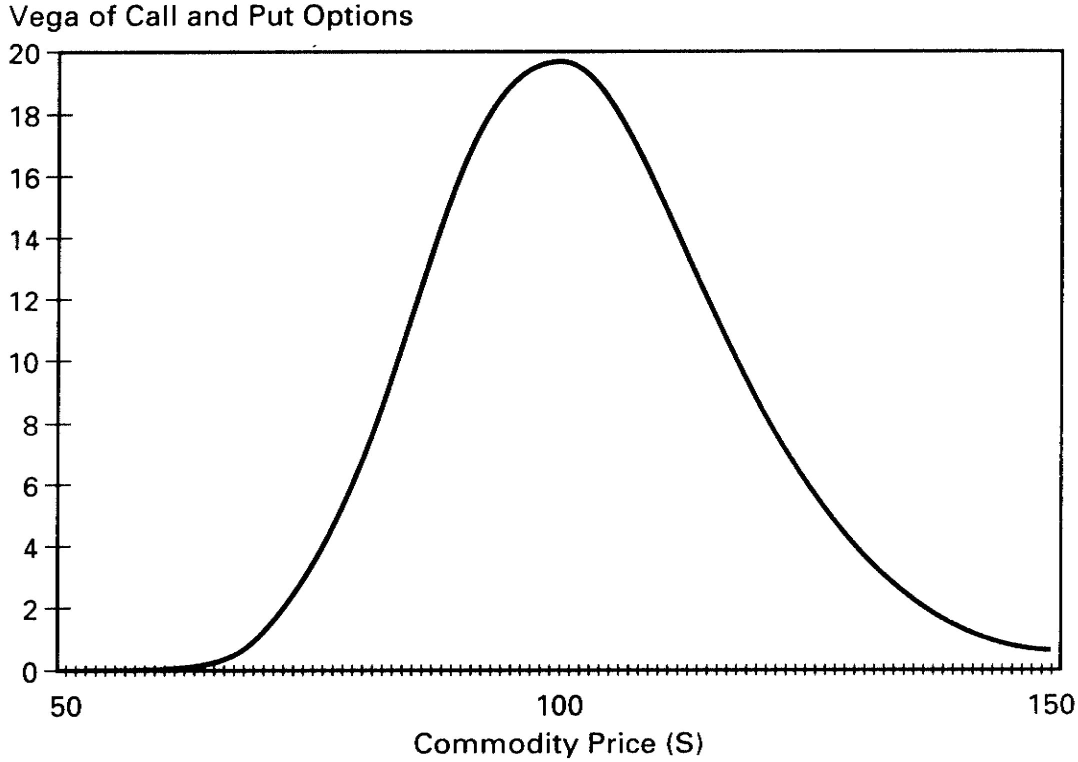
Change in the Time to Expiration

The partial derivative of the option price with respect to the time to expiration parameter is called the option's *theta*. The theta of the call is

$$\begin{aligned}\Theta_c = \frac{\partial c}{\partial T} &= Se^{(b-r)T}n(d_1)\frac{\sigma}{2\sqrt{T}} + (b-r)Se^{(b-r)T}N(d_1)\\ &+ rXe^{-rT}N(d_2) \lesseqgtr 0.\end{aligned} \tag{11.36a}$$

The expression shows that the sensitivity of call option price to changes in the time to expiration of the option is the sum of three components. The first term on the right-hand side is positive and reflects the increase in option price due to the fact that an increase in the time to expiration increases the probability of upward price movements in the commodity price and, hence, increases the value of the option. The second term may be positive or negative depending on whether the cost-of-carry rate, b, is greater than or less than the interest rate, r. If $b > r$, the term is positive since as the time to expiration increases the present value of the expected terminal commodity price grows large (recall that the underlying commodity price grows at rate b while the discount rate of the terminal value of the option is r). Finally, the third term is positive. As time to expiration increases, the present value

FIGURE 11.9 European option vega as a function of the underlying commodity price (S). The commodity price range is from 50 through 150. The options have exercise price (X) of 100 and time to expiration (T) of three months. The cost-of-carry rate (b) is 8 percent, and the riskless rate of interest (r) is 8 percent. The standard deviation of the logarithm of the commodity price ratios (σ) is 30 percent.



of the exercise price grows small. Note that the only case where the overall value of theta is unambiguously positive is when $b \geqslant r$.

For the call option in Example 11.7, $b < r$, so we know that theta need not be positive. The value of theta is, nonetheless, positive at 3.6927. In other words, the option price increases as the time to expiration increases. To see the origin of this result, we examine the values of each of the three terms in the derivative: 4.6457, −2.3446 and 1.3916. The largest component of the call option's theta in this illustration, 4.6457, comes from the increased probability of large commodity price movements. Because $b < r$, the second term is negative. As the time to expiration increases, the value of the call option falls because the commodity price is expected to increase at a lower rate than the riskless rate of interest. The value of this component is −2.3466. Finally, the value of the third term is 1.3916, indicating that the call option value increases because the present value of the exercise price is reduced as the time to expiration is increased.

Theta provides information on the decay in option value as the time to expiration elapses. The theta of the call option in Example 11.7 is 3.6927. This implies that the *time decay* of this option is 3.6927 × 1/365 or slightly over one cent over the next day and 3.6927 × 7/365 or about seven cents over the next week, holding other factors constant.

The theta of the European put option is

$$\Theta_p = \frac{\partial p}{\partial T} = Se^{(b-r)T} n(d_1) \frac{\sigma}{2\sqrt{T}} - (b-r)Se^{(b-r)T} N(-d_1) - rXe^{-rT} N(-d_2) \lesseqgtr 0. \tag{11.36b}$$

The interpretation of the terms in the expression of the put option's theta parallel those of the call option. The first term is the increase in put value resulting from the prospect of larger commodity price movements when the time to expiration is longer. The second term is negative if $b > r$. In the case of the put, option value increases when the cost-of-carry rate is below the interest rate. The third term is positive. It reflects the fact that an increase in the time to expiration delays the receipt of the exercise price and hence reduces the put option value. The value of the theta for the put option in Exercise 11.8 is 5.2143, with the individual components of the sum being 4.6457, 2.3136, and −1.7450.

11.6 EUROPEAN EXCHANGE OPTION

Closely related to the European commodity options with a fixed exercise price are European options that entitle the holder to exchange one commodity for another. Such options are commonplace, although they are usually embedded within some other contract. In Chapter 8, for example, we discussed the fact that the T-bond futures contract permits delivery of any of a number of eligible T-bond issues and that the short will deliver the cheapest. The short, in this instance, has a exchange option that permits him to exchange the T-bond bond he presently holds for a cheaper issue, should a cheaper issue become available. Many agricultural futures contracts also have such an *exchange option* or *quality option* embedded in their contract design.

The derivation of the exchange option formula can follow the same risk-neutral valuation approach that was used earlier in the chapter, so the approach will not be repeated here. The *valuation equation of a European exchange option* that permits its holder to exchange commodity 2 for commodity 1, that is, to "buy" commodity 1 with commodity 2, is

$$c(S_1, T; S_2) = S_1 e^{(b_1 - r)T} N(d_1) - S_2 e^{(b_2 - r)T} N(d_2), \tag{11.37}$$

where

$$d_1 = \frac{\ln(S_1/S_2) + (b_1 - b_2 + .5\sigma^2)T}{\sigma\sqrt{T}}, \tag{11.37a}$$

$$d_2 = d_1 - \sigma\sqrt{T}, \tag{11.37b}$$

and

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2. \tag{11.37c}$$

Variables subscripted with 1 apply to commodity 1, and variables subscripted with 2 apply to commodity 2. The meaning of each variable is described earlier in the chapter.

An important observation regarding (11.37) is that the call option formula described earlier in this chapter is simply a special case of this valuation equation. Suppose we allow commodity 2 to be the riskless asset. The current price of commodity 2, S_2, is, therefore, Xe^{-rT}, the cost-of-carry rate, b_2, is the riskless rate of interest, r, and the standard deviation of return, σ_2, equals zero. With these substitutions, equation (11.37) becomes the European call option formula (11.25).

Another important observation regarding (11.39) is that the value of a call option to "buy" commodity 1 with commodity 2, $c(S_1, T; S_2)$, equals the value of a put option to "sell" commodity 2 for commodity 1, $p(S_2, T; S_1)$. In the case of the call, the option is exercised at expiration if the proceeds $S_{1,T} - S_{2,T} > 0$, that is, if the terminal price of commodity 1 exceeds the terminal price of commodity 2; otherwise, it is not exercised. In the case of the put, the option is exercised at expiration if the proceeds $S_{1,T} - S_{2,T} > 0$; that is, if commodity 2 is cheaper than commodity 1; otherwise, it is not exercised. But the structure of these two valuation problems is identical, so

$$c(S_1, T; S_2) = p(S_2, T; S_1). \tag{11.38}$$

Returning to the T-bond futures contract specification, recall that at the end of Chapter 8, we argued that the futures price equals the price of the cheapest-to-deliver bond less the value of the quality option. If the T-bond futures contract has only two T-bond issues eligible for delivery, the valuation formula (11.38) can be used to value the quality option. With more eligible issues, the model must be generalized.[9]

EXAMPLE 11.9

Suppose that there are two bonds eligible for delivery on the T-bond futures contract. The time to expiration of the futures is three months. Bond 1 is currently the cheapest to deliver. Its price is 99 and its coupon is 6 percent. Bond 2 is priced at

[9]The exchange option formula for the two-asset case where both assets have a cost-of-carry rate equal to the riskless rate of interest was derived by Margrabe (1978). The formula presented here generalizes the Margrabe result to allow the assets to have different carry rates. The n-asset exchange option was later developed by Margrabe (1982).

102, and its coupon is 9 percent. The standard deviation of the continuously compounded return is 15 percent for bond 1 and 12 percent for bond 2. The correlation between their rates of return is .9. The riskless rate of interest is 7 percent. Compute the value of the exchange option. For the sake of simplicity, assume that both bonds have conversion factors equal to one and that coupon interest is paid continuously over the futures contract's remaining life.

$$p(B_2, T; B_1) = 99e^{(.07-.06-.07).25}N(d_1) - 102e^{(.07-.09-.07).25}N(d_2),$$

where

$$d_1 = \frac{\ln(99/102) + (.07 - .06 - .07 + .09 + .5\sigma^2).25}{\sigma\sqrt{.25}},$$

$$d_2 = d_1 - \sigma\sqrt{.25},$$

and

$$\sigma^2 = .15^2 + .12^2 - 2 \times .9 \times .15 \times .12 = .0045.$$

Substituting $\sigma = .0671$ into the expressions for d_1 and d_2, and then the values $d_1 = -.6495$ and $d_2 = -.6830$ into the option formula shows that the value of the exchange option is .50.

11.7 VALUATION OF AMERICAN OPTIONS

We noted in Chapter 10 that the American option is worth at least as much as the European option because of the fact the American option may be exercised early. The value of the American call and put can, therefore, be written as

$$C(S, T; X) = c(S, T; X) + \epsilon_C(S, T; X), \tag{10.9a}$$

and

$$P(S, T; X) = p(S, T; X) + \epsilon_P(S, T; X). \tag{10.9b}$$

The value of the early exercise privilege, $\epsilon_C(S,T;X)$, depends on the relation between the cost-of-carry rate, b, and the riskless rate, r.

In the case of the call option, the early exercise privilege has value only if $b < r$. In this case, the cost of carrying the underlying commodity is less than the cost of funds tied up in the commodity. As a result, it may be beneficial to exercise a call option and take possession of the commodity because the earnings on the commodity exceed the cost of funds tied up in the commodity. For example, it may be desirable to exercise a call option on a foreign currency to earn the interest on the foreign currency if the foreign interest rate exceeds the U.S. rate. If $b \geqslant r$, early exercise of a call is not optimal because there is a cost to holding the commodity. By continuing to hold the call option, all the potential price gains achievable from holding the commodity are also achievable, and the cost of holding the commodity is avoided. For example, it is never optimal to exercise early an option on a stock that does not pay dividends (i.e., $b = r$).

In the case of the put option, early exercise is always a possibility. Intuitively, early exercise is desirable if the profit from the put option is sufficiently large so that the interest that could be earned by investing the profit now exceeds the possibility of an even greater profit from continuing to hold the put.

Explicit analytical solutions for the price of American options are unknown. If the American option will not be exercised early, the European option formula holds. But if early exercise could be desirable, the American option value exceeds the European value by an amount (frequently quite small) that can only be approximated by numerical techniques. Two approximation techniques that are commonly applied in practice are the binomial method of Cox, Ross, and Rubinstein (1979), and the quadratic approximation method of Barone–Adesi and Whaley (1987). The binomial method is used in Chapter 13 to value a put on a dividend-paying stock, and the quadratic approximation is used in Chapter 14 to value stock index and stock index futures options.

11.8 ESTIMATION OF THE OPTION PRICING PARAMETERS

The European option pricing models (11.25) and (11.28) and the American option approximation methods to be discussed in Chapters 13 and 14 are, in general, very easy to use. The exercise price, X, and the time to expiration, T, are terms of the option contract. The commodity price, S, the riskless rate of interest, r, and the cost-of-carry rate, b, are easily accessible, market-determined values.[10] The most difficult parameter to estimate (and, for that matter, the parameter estimate about which investors most commonly disagree) is the standard deviation of the rate of return of the underlying commodity. In general, two methods are used—historical volatility estimation and implied volatility estimation.

[10] As a proxy for the riskless interest rate, T-bill rates are typically used. Recall from footnote 2 in Chapter 8 that the continuously compounded, effective annual interest rate, r, is obtained by computing

$$r = \frac{\ln(100/B_d)}{T},$$

where B_d is the price of the T-bill (e.g., 100 less the average of the T-bill's bid and ask discounts) and T is the time to maturity of the T-bill expressed in years.

Historical Volatility Estimation

Earlier in this chapter, we assumed that the mean and the standard deviation of the continuously compounded rates of return, $R_t = \ln(S_t/S_{t-1})$, are constant through time. The volatility parameter in the option pricing formula is the future volatility rate of the commodity. If the volatility parameter is stationary through time, however, we can use past returns to estimate historical volatility and then use historical volatility as the estimate of future volatility.

The estimator most commonly used for calculating the variance of the rate of return on the commodity, σ_h^2, is

$$\hat{\sigma}_h^2 = \frac{1}{T-1}\sum_{t=1}^{T}(R_t - \hat{\mu})^2, \tag{11.39}$$

where T is the number of time series return observations used in the estimation,[11] R_t is the continuously compounded rate of return on the commodity in month t [i.e., $\ln(S_t/S_{t-1})$], and $\hat{\mu}$ is the estimate of the mean rate of return,

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} R_t. \tag{11.40}$$

An estimate of the standard deviation of the rate of return on the commodity can be obtained by taking the square root of the variance estimate, that is, $\hat{\sigma}_h = \sqrt{\hat{\sigma}_h^2}$.

The rates of return used in equations (11.39) and (11.40) may be for any length period—a day, a week, or a month. In general, the shorter the distancing interval between price observations, the better since more information goes into the estimate. So weekly returns are certainly superior to monthly returns in the estimation of volatility, holding the overall length of the estimation period constant.

Following the same logic, it would seem daily returns are a better source of information than weekly returns. This is generally not the case, however. Stock returns, for example, demonstrate seasonality by day of week.[12] In general, stock returns on Fridays are significantly higher than average, and those on Mondays are significantly lower than average. Other commodities also have day-of-the-week seasonality, but the seasonality has a different structure. Furthermore, independent of the underlying commodity, using daily data forces the investigator to decide how weekend returns should be accounted for. Should the rate of return from Friday close to Monday close be counted as a 3-day rate of return (calendar days) or a 1-day rate of return (trading days)? Because of the empirical anomalies associated with daily returns and because the issue of how to handle weekend returns has not been satisfactorily resolved, weekly returns are probably the best bet when it comes to estimating the historical volatility of commodity returns.

Another issue that arises when using the historical estimator has to do with

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[11] Note that $T + 1$ price observations are needed to generate T rates of return.

[12] See, for example, French (1980) and Gibbons and Hess (1981).

how many return observations to use in the computation of the volatility. On one hand, the more information that is used in the estimation process, the more precise that estimate becomes. On the other hand, the longer the time period over which volatility is estimated, the greater the likelihood that the stationarity assumption will be violated, in which case, formula (11.39) is no longer an unbiased estimate of the commodity's rate-of-return variance. In the absence of information indicating that the stationarity assumption has been violated, twenty-six weeks of return observations are probably enough to ensure reasonably accurate volatility estimation.

The volatility estimate computed using equation (11.39) computes the variance of the rate of return for the length of time between the price observations used to compute the rates of return. Thus, if weekly returns are used, the variance estimate from (11.39) is the variance of the rate of return over a week. To annualize this value, we have to multiply the variance by the number of weeks in the year, that is, $\sigma_{ha}^2 = 52\sigma_{hw}^2$, where the subscripts a and w denote annual and weekly, respectively. The transformation for annualizing the weekly standard deviation is therefore $\sigma_{ha} = \sqrt{52}\sigma_{hw}$.[13]

One final issue with respect to historical volatility estimation is worth noting. The estimators shown above generally use close-to-close price information when generating the rates of return. This has been accepted as common practice since, traditionally, the recorded histories of commodity prices are prices reported for the last transaction of the day. With the advent of sophisticated computer and database technologies, it has now become easier to record and maintain larger information sets, with most commodity and option exchanges now recording and maintaining transaction price files. This more refined information allows more precise estimation of volatility. For example, Parkinson (1980) and Garman and Klass (1980) develop alternative estimators of variance that use open, high, low, and close commodity prices and show that these estimators are eight times "better" than the traditional estimator (11.39).

Implied Volatility Estimation

An alternative volatility estimation procedure arises from the option pricing model itself. Since all of the parameters of the option pricing model, except σ, are known or can be estimated with little uncertainty, one needs only to equate the observed market price of the option with its formula value, that is,

$$V_j = \hat{V}_j(\sigma_j), \tag{11.41}$$

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[13]To transform a volatility estimate computed using daily returns to an annual volatility, the daily estimate is usually multiplied by the square root of the number of trading days in the year (typically, $\sqrt{253}$), rather than the number of calendar days in the year ($\sqrt{365}$). The motivation for this adjustment is that studies of daily stock returns indicate that the volatility of return from Friday close to Monday close (three days) is about the same as the volatility from close-to-close during any other pair of adjacent trading days (one day). See, for example, Stoll and Whaley (1990a). Thus, treating weekends like a single trading day provides the most appropriate adjustment for daily stock return volatilities. The empirical evidence regarding weekend volatility in non-stock markets, however, is scant, so the generality of this result to other commodities is not known. For non-stock markets, a safer procedure may be to use weekly returns, as was noted earlier in this section.

where V is the observed price of the option, $\hat{V}$ is the model price of the option, and solve for σ. An analytical expression for the variance parameter cannot be derived; however, accurate approximation is possible through "trial-and-error," in much the same manner as one solves for a yield to maturity on a bond.

Volatilities computed in this manner are called "implied volatilities" or "implied standard deviations." They may be interpreted as the market consensus volatility in the sense that the market price of the option is used to impute the volatility estimate.

If one considers all of the options written on a given commodity, it would seem reasonable to believe that they will all yield the same estimate of volatility on the underlying commodity. This is not the case, however. There are a variety of reasons to cause the estimates to be different.

Non-Simultaneity of Prices. Estimating the implied volatility using (11.41) assumes that the option price and the commodity price are observed at the same instant in time. Frequently, it is the case that the only information that is available is the option and commodity prices at the times at which they were last traded. It is unlikely that these trades, one in the option market and one in the commodity market, occurred at the same instant, and, to the extent that they are not contemporaneous, there will be error in the estimate of volatility.

Bid-Ask Prices. Even if the option and commodity price observations used in (11.41) are simultaneous, there is a problem with what the prices represent. If markets were perfectly liquid and frictionless, trades would clear at the equilibrium price of the security. Neither descriptor is true, however. Market makers provide market liquidity by standing ready to immediately buy or sell securities. Since market makers have capital (both investment and human) tied up in their operations, they demand a rate of return on their capital, which they extract by setting the bid price of a security below the ask price. When market orders are executed, therefore, they are at the bid or the ask, depending upon whether the individual entering the market to trade wanted to sell or buy. Since there is no way of discerning the motivation of the trader who was involved in the last observed transaction, implied volatility estimates have error when the bid price of the option is matched with the ask price of the commodity and vice versa.

Model Mis-Specification. Using (11.41) to estimate volatility is also subject to model mis-specification. The technique assumes that the option pricing model used for $\hat{V}_j(\sigma_j)$ is correctly specified. If it is not, then there is obviously going to be error in the estimate of the standard deviation of the rate of return on the commodity. Model assumptions that could be violated, for example, are the assumption of log normality of stock prices or independence of returns.

To mitigate the problems associated with using a single implied volatility estimate to represent the volatility of the underlying commodity, the implied volatilities for several options on the same commodity are averaged to form an overall estimate. The nature of the averaging schemes vary, so it is best to begin with a general statement of the average implied volatility, that is,

$$\hat{\sigma} = \sum_{j=1}^{n} \omega_j \hat{\sigma}_j \Big/ \sum_{j=1}^{n} \omega_j, \tag{11.42}$$

where ω_j is the weight applied to the jth estimate of volatility and n is the number of options for which volatility estimates were obtained.

The particular weighting schemes used in the literature have been many and varied. Schmalensee and Trippi (1978) and Patell and Wolfson (1979), for example, use an equal weighted average, $\omega_j = 1/n$, $j = 1,...,n$. Their motivation for doing so is that each volatility estimate is equally valuable in the determination of the overall volatility for the commodity. Latane and Rendleman (1976), on the other hand, weight according to the partial derivative of the call price with respect to the standard deviation of the commodity return, that is, $\partial V_j/\partial\hat{\sigma}_j$, $j = 1,...,n$. In doing so, the standard deviation estimates of options that are theoretically more sensitive to the value of σ are weighted more heavily than those that are not. Chiras and Manaster (1978) follow a similar logic in using the elasticity of the call price with respect to standard deviation, $\partial V_j\hat{\sigma}_j/\partial\hat{\sigma}_j V_j$, $j = 1,...,n$. Unfortunately, their scheme is seriously flawed. Using elasticity as the basis of the weighting scheme implies that volatility estimates for out-of-the-money options receive the highest weight. Out-of-the-money options generally do not produce very accurate volatility estimates because the markets for these options are relatively illiquid (inducing serious nonsimultaneity problems) and the options themselves have high bid-ask spreads (inducing bid-ask errors). Finally, Whaley (1982) uses nonlinear regression to estimate one value of σ using all of the option pricing information simultaneously, that is,

$$V_j = \hat{V}_j(\sigma) + \epsilon_j. \tag{11.43}$$

The properties of the maximum likelihood estimator from (11.43) are, perhaps, the best understood of the available alternatives.

Regardless of the weighting scheme, however, there appears to be strong empirical support in favor of an implied volatility measure. Latane and Rendleman and Chiras and Manaster correlate the historical and implied measures on the actual standard deviation of return and conclude that the implied estimate is a markedly superior predictor. The market apparently uses more information than merely an historical estimate in assessing the commodity's expected volatility.

11.9 SUMMARY

In this chapter, option pricing equations have been derived in detail for European options on different types of underlying assets. The chapter begins with an intuitive discussion of the risk-neutral valuation approach used in deriving option pricing formulas. Next the price and return distributions assumed for commodities are described. In section 3, risk-neutral valuation of a European option is carried out in detail, and variations of the basic valuation equation for different types of underlying commodities are shown. Using put-call parity, put valuation equations are then derived.

The price of an option on a commodity depends on the spot price of the commodity, the exercise price of the option, the cost of carry of the commodity, the riskless rate, the standard deviation of the return of the commodity, and the time to maturity of the option. In section 5, the effect of changes in each of these variables on the option price is analyzed.

Section 6 presents the valuation equation for an option that permits its holder to exchange one risky commodity for another. This option, called an exchange option, is embedded in many types of futures contracts and was introduced in earlier chapters. Valuing the delivery option embedded in the T-bond futures contract is used as an illustration.

The factors underlying the value of an American option are the same as those underlying a European option except that the American option has the additional benefit of early exercise. Section 7 names two popular approaches for valuing American options. The methods are described in detail in Chapters 13 and 14.

In practice, the most important variable affecting the price of an option is the volatility of the underlying commodity. Section 8 explains the two approaches for estimating volatility—historical volatility and implied volatility.

APPENDIX 11.1

Proof that $E(\tilde{S}_T) = S_0 e^{\alpha T} = S_0 e^{(\mu+\sigma^2/2)T}$,
where μ *and* σ^2 *are the mean and the variance*
of the normally distributed continuously compounded rate of return

Begin by rewriting the expected terminal price as the expected price relative,

$$E(\tilde{S}_T/S_0) = e^{\alpha T} = E(e^{\tilde{x}}), \tag{A1.1}$$

where $\tilde{x}$ is the normally distributed, continuously compounded rate of return from 0 through T. $\tilde{x}$ can be reexpressed in terms of μ, σ, and the unit normally distributed variable z. Using (11.10),

$$\tilde{S}_T/S_0 = e^{\tilde{x}} = e^{\mu T + \sigma\sqrt{T}\tilde{z}}.$$

Substituting this result into (A1.1),

$$\begin{aligned} e^{\alpha T} &= E(e^{\mu T + \sigma\sqrt{T}\tilde{z}}) \\ &= e^{\mu T} E(e^{\sigma\sqrt{T}\tilde{z}}). \end{aligned} \tag{A1.2}$$

The term $E(e^{\sigma\sqrt{T}\tilde{z}})$ in (A1.2) may be simplified as follows:

$$\begin{aligned} E(e^{\sigma\sqrt{T}\tilde{z}}) &= \int_{-\infty}^{+\infty} e^{\sigma\sqrt{T}z} e^{-z^2/2} \frac{1}{\sqrt{2\pi}} dz \\ &= \int_{-\infty}^{+\infty} e^{\sigma\sqrt{T}z - z^2/2} \frac{1}{\sqrt{2\pi}} dz \\ &= \int_{-\infty}^{+\infty} e^{\sigma^2 T/2 - \sigma^2 T/2 + \sigma\sqrt{T}z - z^2/2} \frac{1}{\sqrt{2\pi}} dz \\ &= e^{\sigma^2 T/2} \int_{-\infty}^{+\infty} e^{-\sigma^2 T/2 + \sigma\sqrt{T}z - z^2/2} \frac{1}{\sqrt{2\pi}} dz \\ &= e^{\sigma^2 T/2} \int_{-\infty}^{+\infty} e^{-(\sigma\sqrt{T} - z)^2/2} \frac{1}{\sqrt{2\pi}} dz \\ &= e^{\sigma^2 T/2}. \end{aligned} \tag{A1.3}$$

Substituting (A1.3) into (A1.2), taking the logarithm of both sides, and then factoring T gives

$$\alpha = \mu + \sigma^2/2. \tag{A1.4}$$

APPENDIX 11.2

APPROXIMATIONS FOR THE CUMULATIVE NORMAL DENSITY FUNCTION

The probability that a drawing from a unit normal distribution will produce a value less than the constant d is

$$\text{Prob}(\tilde{z} < d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = N(d).$$

Below are two polynomials that provide reasonably accurate approximations for the above integral.

Approximation 1

$$N(d) \approx 1 - a_0 e^{-d^2/2}(a_1 t + a_2 t^2 + a_3 t^3),$$

where

$$t = 1/(1 + 0.33267d) \qquad \begin{array}{ll} a_0 = 0.3989423 & a_1 = 0.4361836 \\ a_2 = -0.1201676 & a_3 = 0.9372980 \end{array}$$

With this approximation method, the value of d must be greater than or equal to 0. The maximum absolute error of this approximation method is 0.00001.

Approximation 2

$$N(d) \approx 1 - a_0 e^{-d^2/2}(a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5),$$

where

$$t = 1/(1 + 0.2316419d) \qquad \begin{array}{ll} a_0 = 0.3989423 & a_1 = 0.319381530 \\ a_2 = -0.356563782 & a_3 = 1.781477937 \\ a_4 = -1.821255978 & a_5 = 1.330274429 \end{array}$$

With this approximation method, the value of d must be greater than or equal to 0. The maximum absolute error of this approximation method is 0.000000075.

APPENDIX 11.3

CUMULATIVE NORMAL PROBABILITY TABLES

The probability that a drawing from a unit normal distribution will produce a value less than the constant d is

$$\text{Prob}(\tilde{z} < d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = N(d).$$

Range of d: $-4.99 \leq d \leq -2.50$

d	-0.00	-0.01	-0.02	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.09
-4.90	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-4.80	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-4.70	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-4.60	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-4.50	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-4.40	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
-4.30	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
-4.20	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
-4.10	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002	0.00001	0.00001
-4.00	0.00003	0.00003	0.00003	0.00003	0.00003	0.00003	0.00002	0.00002	0.00002	0.00002
-3.90	0.00005	0.00005	0.00004	0.00004	0.00004	0.00004	0.00004	0.00004	0.00003	0.00003
-3.80	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005
-3.70	0.00011	0.00010	0.00010	0.00010	0.00009	0.00009	0.00008	0.00008	0.00008	0.00008
-3.60	0.00016	0.00015	0.00015	0.00014	0.00014	0.00013	0.00013	0.00012	0.00012	0.00011
-3.50	0.00023	0.00022	0.00022	0.00021	0.00020	0.00019	0.00019	0.00018	0.00017	0.00017
-3.40	0.00034	0.00032	0.00031	0.00030	0.00029	0.00028	0.00027	0.00026	0.00025	0.00024
-3.30	0.00048	0.00047	0.00045	0.00043	0.00042	0.00040	0.00039	0.00038	0.00036	0.00035
-3.20	0.00069	0.00066	0.00064	0.00062	0.00060	0.00058	0.00056	0.00054	0.00052	0.00050
-3.10	0.00097	0.00094	0.00090	0.00087	0.00084	0.00082	0.00079	0.00076	0.00074	0.00071
-3.00	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.00100
-2.90	0.00187	0.00181	0.00175	0.00169	0.00164	0.00159	0.00154	0.00149	0.00144	0.00139
-2.80	0.00256	0.00248	0.00240	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
-2.70	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.00280	0.00272	0.00264
-2.60	0.00466	0.00453	0.00440	0.00427	0.00415	0.00402	0.00391	0.00379	0.00368	0.00357
-2.50	0.00621	0.00604	0.00587	0.00570	0.00554	0.00539	0.00523	0.00508	0.00494	0.00480

CUMULATIVE NORMAL PROBABILITY TABLES

The probability that a drawing from a unit normal distribution will produce a value less than the constant d is

$$\text{Prob}(\tilde{z} < d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = N(d).$$

Range of d: $-2.49 \leq d \leq 0.00$

d	-0.00	-0.01	-0.02	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.09
-2.40	0.00820	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
-2.30	0.01072	0.01044	0.01017	0.00990	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
-2.20	0.01390	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.01160	0.01130	0.01101
-2.10	0.01786	0.01743	0.01700	0.01659	0.01618	0.01578	0.01539	0.01500	0.01463	0.01426
-2.00	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.01970	0.01923	0.01876	0.01831
-1.90	0.02872	0.02807	0.02743	0.02680	0.02619	0.02559	0.02500	0.02442	0.02385	0.02330
-1.80	0.03593	0.03515	0.03438	0.03362	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
-1.70	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.03920	0.03836	0.03754	0.03673
-1.60	0.05480	0.05370	0.05262	0.05155	0.05050	0.04947	0.04846	0.04746	0.04648	0.04551
-1.50	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
-1.40	0.08076	0.07927	0.07780	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
-1.30	0.09680	0.09510	0.09342	0.09176	0.09012	0.08851	0.08691	0.08534	0.08379	0.08226
-1.20	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10383	0.10204	0.10027	0.09853
-1.10	0.13567	0.13350	0.13136	0.12924	0.12714	0.12507	0.12302	0.12100	0.11900	0.11702
-1.00	0.15866	0.15625	0.15386	0.15150	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
-0.90	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
-0.80	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
-0.70	0.24196	0.23885	0.23576	0.23270	0.22965	0.22663	0.22363	0.22065	0.21770	0.21476
-0.60	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.24510
-0.50	0.30854	0.30503	0.30153	0.29806	0.29460	0.29116	0.28774	0.28434	0.28096	0.27760
-0.40	0.34458	0.34090	0.33724	0.33360	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
-0.30	0.38209	0.37828	0.37448	0.37070	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
-0.20	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
-0.10	0.46017	0.45620	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
0.00	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414

CUMULATIVE NORMAL PROBABILITY TABLES

The probability that a drawing from a unit normal distribution will produce a value less than the constant d is

$$\text{Prob}(\tilde{z} < d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = N(d).$$

Range of d: $0.00 \leq d \leq 2.49$

d	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.10	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.20	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.30	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.40	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.50	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.60	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.70	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.80	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.90	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.00	0.84134	0.84375	0.84614	0.84850	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.10	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.20	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.30	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.40	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.50	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.60	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.70	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.80	0.96407	0.96485	0.96562	0.96637	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.90	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.00	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.10	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.20	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.30	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.40	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361

CUMULATIVE NORMAL PROBABILITY TABLES

The probability that a drawing from a unit normal distribution will produce a value less than the constant d is

$$\text{Prob}(\tilde{z} < d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = N(d).$$

Range of d: $2.50 \le d \le 4.99$

d	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.50	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.60	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.70	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.80	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.90	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.00	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99897	0.99900
3.10	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.20	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.30	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.40	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.50	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.60	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.70	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.80	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.90	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.00	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.10	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999
4.20	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999
4.30	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999
4.40	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.50	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.60	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.70	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.80	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
4.90	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

APPENDIX 11.4

PARTIAL DERIVATIVES OF EUROPEAN COMMODITY OPTION VALUATION EQUATIONS

The valuation equations for the European call and put options are

$$c(S,T;X) = Se^{(b-r)T}N(d_1) - Xe^{-rT}N(d_2) \tag{A4.1}$$

and

$$p(S,T;X) = Xe^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1), \tag{A4.2}$$

respectively, where

$$d_1 = \frac{\ln(S/X) + (b + .5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{A4.3}$$

$$d_2^2 = d_1^2 - 2d_1\sigma\sqrt{T} + \sigma^2 T \tag{A4.4}$$

$$\begin{aligned} d_2^2 &= d_1^2 - 2[\ln(S/X) + bT + .5\sigma^2 T] + \sigma^2 T \\ &= d_1^2 - 2\ln(Se^{bT}/X) \end{aligned} \tag{A4.5}$$

$$n(d_2) = \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \tag{A4.6}$$

$$\begin{aligned} n(d_2) &= \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2 + 2\ln(Se^{bT}/X)} \\ &= \frac{1}{\sqrt{2\pi}} e^{-d_1^2} e^{\ln(Se^{bT}/X)} \\ &= n(d_1)Se^{bT}/X \end{aligned} \tag{A4.7}$$

$$n(d_1) = n(d_2)X/Se^{bT} \tag{A4.8}$$

$$\Delta_c = \frac{\partial c}{\partial S} = e^{(b-r)T}N(d_1) + Se^{(b-r)T}\frac{\partial N(d_1)}{\partial S} - Xe^{-rT}\frac{\partial N(d_2)}{\partial S}$$
$$= e^{(b-r)T}N(d_1) + Se^{(b-r)T}\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Xe^{-rT}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S}$$
$$= e^{(b-r)T}N(d_1) + Se^{(b-r)T}n(d_1)\frac{\partial d_1}{\partial S} - Xe^{-rT}n(d_2)\frac{\partial d_2}{\partial S}$$
$$= e^{(b-r)T}N(d_1) + Se^{(b-r)T}n(d_1)\frac{\partial d_1}{\partial S} - Xe^{-rT}n(d_1)Se^{bT}/X\frac{\partial d_1}{\partial S}$$
$$= e^{(b-r)T}N(d_1) > 0 \qquad \textbf{(A4.9a)}$$

$$\Delta_p = \frac{\partial p}{\partial S} = Xe^{-rT}\frac{\partial N(-d_2)}{\partial S} - e^{(b-r)T}N(-d_1) - Se^{(b-r)T}\frac{\partial N(-d_1)}{\partial S}$$
$$= -e^{(b-r)T}N(-d_1) - Xe^{-rT}n(-d_1)Se^{bT}/X\frac{\partial d_1}{\partial S} + Se^{(b-r)T}n(-d_1)\frac{\partial d_1}{\partial S}$$
$$= -e^{(b-r)T}N(-d_1) < 0 \qquad \textbf{(A4.9b)}$$

$$\eta_c = \frac{\partial c/c}{\partial S/S} = \Delta_c\frac{S}{c} = e^{(b-r)T}N(d_1)\frac{S}{c} > 1 \qquad \textbf{(A4.10a)}$$

$$\eta_p = \frac{\partial p/p}{\partial S/S} = \Delta_p\frac{S}{p} = -e^{(b-r)T}N(-d_1)\frac{S}{p} < -1 \qquad \textbf{(A4.10b)}$$

$$\gamma_c = \frac{\partial \Delta_c}{\partial S} = \frac{\partial e^{(b-r)T}N(d_1)}{\partial S} = e^{(b-r)T}n(d_1)\frac{\partial d_1}{\partial S} = \frac{e^{(b-r)T}n(d_1)}{S\sigma\sqrt{T}} > 0 \qquad \textbf{(A4.11a)}$$

$$\gamma_p = \frac{\partial \Delta_p}{\partial S} = \frac{-\partial e^{(b-r)T}N(-d_1)}{\partial S} = \frac{e^{(b-r)T}n(d_1)}{S\sigma\sqrt{T}} = \gamma_c > 0 \qquad \textbf{(A4.11b)}$$

$$\frac{\partial c}{\partial X} = -e^{-rT}N(d_2) < 0 \qquad \textbf{(A4.12a)}$$

$$\frac{\partial p}{\partial X} = e^{-rT}N(-d_2) > 0 \qquad \textbf{(A4.12b)}$$

$$\frac{\partial c}{\partial b} = TSe^{(b-r)T}N(d_1) + Se^{(b-r)T}\frac{\partial N(d_1)}{\partial b} - Xe^{-rT}\frac{\partial N(d_2)}{\partial b} \qquad \textbf{(A4.13a)}$$
$$= TSe^{(b-r)T}N(d_1) > 0$$

$$\frac{\partial p}{\partial b} = Xe^{-rT}\frac{\partial N(-d_2)}{\partial b} - TSe^{(b-r)T}N(-d_1) - Se^{(b-r)T}\frac{\partial N(-d_1)}{\partial b} \qquad \textbf{(A4.13b)}$$
$$= -TSe^{(b-r)T}N(-d_1) < 0$$

$$\begin{aligned}\frac{\partial c}{\partial r} &= TSe^{(b-r)T}N(d_1) - TSe^{(b-r)T}N(d_1) + Se^{(b-r)T}\frac{\partial N(d_1)}{\partial r}\\ &\quad + TXe^{-rT}N(d_2) - Xe^{-rT}\frac{\partial N(d_2)}{\partial r}\\ &= TXe^{-rT}N(d_2) > 0 \end{aligned} \tag{A4.14a}$$

$$\begin{aligned}\frac{\partial p}{\partial r} &= -TXe^{-rT}N(-d_2) + Xe^{-rT}\frac{\partial N(-d_2)}{\partial r}\\ &\quad - TSe^{(b-r)T}N(-d_1) + TSe^{(b-r)T}N(-d_1) - Se^{(b-r)T}\frac{\partial N(-d_1)}{\partial r}\\ &= -TXe^{-rT}N(-d_2) < 0 \end{aligned} \tag{A4.14b}$$

$$\begin{aligned}\text{Vega}_c = \frac{\partial c}{\partial \sigma} &= Se^{(b-r)T}\frac{\partial N(d_1)}{\partial \sigma} - Xe^{-rT}\frac{\partial N(d_2)}{\partial \sigma}\\ &= Se^{(b-r)T}n(d_1)\frac{\partial d_1}{\partial \sigma} - Xe^{-rT}n(d_2)\frac{\partial d_2}{\partial \sigma}\\ &= Se^{(b-r)T}n(d_1)\frac{\partial d_1}{\partial \sigma} - Xe^{-rT}n(d_1)Se^{rT}/X\frac{\partial d_2}{\partial \sigma}\\ &= Se^{(b-r)T}n(d_1)\left[\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right]\\ &= Se^{(b-r)T}n(d_1)\sqrt{T} > 0 \end{aligned} \tag{A4.15a}$$

$$\begin{aligned}\text{Vega}_p = \frac{\partial p}{\partial \sigma} &= Xe^{-rT}\frac{\partial N(-d_2)}{\partial \sigma} - Se^{(b-r)T}\frac{\partial N(-d_1)}{\partial \sigma}\\ &= Xe^{-rT}n(-d_2)\left[-\frac{\partial d_2}{\partial \sigma}\right] - Se^{(b-r)T}n(-d_1)\left[-\frac{\partial d_1}{\partial \sigma}\right]\\ &= Se^{(b-r)T}n(d_1)\left[\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right]\\ &= Se^{(b-r)T}n(d_1)\sqrt{T} > 0 \end{aligned} \tag{A4.15b}$$

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \left[-\frac{\ln(Se^{bT}/X)}{\sigma^2\sqrt{T}} + .5\sqrt{T}\right] - \left[-\frac{\ln(Se^{bT}/X)}{\sigma^2\sqrt{T}} - .5\sqrt{T}\right] = \sqrt{T}$$

$$\begin{aligned}\Theta_c = \frac{\partial c}{\partial T} &= (b-r)Se^{(b-r)T}N(d_1)Se^{(b-r)T}\frac{\partial N(d_1)}{\partial T}\\ &\quad + rXe^{-rT}N(d_2) - Xe^{-rT}\frac{\partial N(d_2)}{\partial T}\\ &= Se^{(b-r)T}n(d_1)\left[\frac{\partial d_1}{\partial T} - \frac{\partial d_2}{\partial T}\right] + (b-r)Se^{(b-r)T}N(d_1) + rXe^{-rT}N(d_2)\\ &= Se^{(b-r)T}n(d_1)\frac{\sigma}{2\sqrt{T}} + (b-r)Se^{(b-r)T}N(d_1) + rXe^{-rT}N(d_2) \lesseqgtr 0 \end{aligned} \tag{A4.16a}$$

$$\Theta_p = \frac{\partial p}{\partial T} = -rXe^{-rT}N(-d_2) + Xe^{-rT}\frac{\partial N(-d_2)}{\partial T}$$
$$- (b-r)Se^{(b-r)T}N(-d_1) - Se^{(b-r)T}\frac{\partial N(-d_1)}{\partial T}$$
$$= -rXe^{-rT}N(-d_2) - (b-r)Se^{(b-r)T}N(-d_1) + Se^{(b-r)T}n(d_1)\left[\frac{\partial d_1}{\partial T} - \frac{\partial d_2}{\partial T}\right]$$
$$= Se^{(b-r)T}n(d_1)\frac{\sigma}{2\sqrt{T}} - (b-r)Se^{(b-r)T}N(-d_1) - rXe^{-rT}N(-d_2) \leq\geq 0 \qquad \textbf{(A4.16b)}$$

$$\frac{\partial d_1}{\partial T} - \frac{\partial d_2}{\partial T} = \left[-\frac{\ln(S/X)}{2\sigma T^{3/2}} + \frac{b}{2\sigma\sqrt{T}} + \frac{\sigma}{4\sqrt{T}}\right] - \left[-\frac{\ln(S/X)}{2\sigma T^{3/2}} + \frac{b}{2\sigma\sqrt{T}} - \frac{\sigma}{4\sqrt{T}}\right] = \frac{\sigma}{2\sqrt{T}}$$