13 COMMON STOCK OPTION CONTRACTS

Common stock option contracts have been traded in the U.S. for many decades. Trading began in the over-the-counter market more than fifty years ago. In April 1973, the Chicago Board Options Exchange (CBOE) became the first organized secondary market in call options on sixteen NYSE common stocks. In 1977, the CBOE introduced put options on stocks. Today, both call and put options are traded on over five hundred different stocks and on five exchanges. In addition to the CBOE, active secondary stock option markets exist on the American Stock Exchange, the Pacific Coast Exchange, the Philadelphia Stock Exchange, and the New York Stock Exchange.

This chapter focuses on stock options. In the first section, the stock option market is described. In the second section, we adapt the arbitrage pricing relations of Chapter 10 to stock option contracts. The principles are modified somewhat to account for the fact that common stocks typically pay discrete dividends during the option's life. In section 3, we value European and American call options on dividend-paying stocks. Even though an American call option may be exercised early, a valuation equation for this option exists. For American put options on stocks, no valuation equation exists. In section 4, the binomial method is used to approximate the value of American put options on dividend-paying stocks. Although the application is specific, the binomial method can be applied to the valuation of virtually any type of option. This method is used again in Chapter 15, for example, to value interest rate options. Finally, in section 5, warrants used to raise new capital are studied. Warrants are like options except that they are issued by a company. If the warrants are exercised, the company's equity may be diluted, something that must be incorporated into warrant valuation.

13.1 COMMON STOCK OPTION MARKETS

In the U.S., stock option contracts are written in denominations of one hundred shares, expire on the Saturday after the third Friday of the contract month, and are American-style. A call option on a stock represents the right to buy one hundred shares of the underlying stock, and a put option represents the right to sell one hundred shares. Although the contract denomination is one hundred shares, the option prices are quoted on a per-share basis. The exercise prices of stock options are in $5 increments above $25 and in $2.50 increments below.

To illustrate these conventions, consider the option prices reported in Table 13.1. The IBM call option with an exercise price of 100 and a January maturity has a price of $3.625. To buy this contract, it would cost $3.625 × 100 = $362.50. The implied terms of this contract are that the buyer has the right to buy one hundred shares of IBM for a total amount of $10,000 at any time between November 13, 1991, and January 17, 1992 (i.e., the third Friday of January).[1]

In reporting stock option prices, a number of conventions are used. In Table 13.1, a number of cells of the table have the entry "r." This means that the particular option did not trade on that day. Other cells have the entry "s," implying that the particular option contract does not exist. Also, beneath the firm's ticker symbol, the closing stock price is reported. This is done for convenience, so the reader does have to turn to the stock market page to find the closing price of the underlying stock.

One final note about Table 13.1 is that the most actively traded stock options on the various exchanges are listed. These are simply the option contracts with the greatest number of transactions on that day. Interspersed among the stock options are usually some stock index options, the subject of the next chapter. In the CBOE active option list, for example, nine of the ten most active options are written on the S&P 100 index.

13.2 PRICE BOUNDS AND ARBITRAGE RELATIONS

The lower price bounds and the put-call parity relations for stock options are presented for non-dividend- and dividend-paying stocks. Without dividends, the arbitrage relations are straightforward since a common stock is a commodity for which the cost-of-carry-rate, b, equals the riskless rate of interest, r. With dividends, the relations are modified slightly to account for the fact that the underlying stock drops in price at the ex-dividend instant, while the exercise price does not.[2] As a result, an anticipated cash dividend generally reduces the value of calls and increases the value of puts.

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[1] For practical purposes, assuming the option expires Friday seems prudent since both the stock market and the option market are closed on Saturday.

[2] The exercise price is, however, adjusted for stock splits and stock dividends. Where the stock split/dividend produces a fractional result, the exercise price is rounded to the nearest eighth.

TABLE 13.1 Stock option contract prices.

LISTED OPTIONS QUOTATIONS

Wednesday, November 13, 1991

Options closing prices. Sales unit usually is 100 shares. Stock close is New York or American exchange final price.

CHICAGO BOARD

Option & NY Close	Strike Price	Calls-Last Nov	Calls-Last Dec	Calls-Last Jan	Puts-Last Nov	Puts-Last Dec	Puts-Last Jan
Alcoa	55	r	s	8 1/4	r	s	r
61 7/8	60	r	3	4	1/16	1 1/16	1 13/16
61 7/8	65	r	7/8	1 3/4	3 1/4	3 1/2	4 1/2
61 7/8	70	r	r	1/2	r	r	8 1/4
61 7/8	75	s	s	1/8	s	s	r
AmGenl	40	r	r	3 7/8	r	r	1/2
43 1/2	45	1/16	r	7/8	r	r	r
Chryslr	10	r	r	3 1/8	r	1/16	1/8
13	12 1/2	9/16	3/4	1 1/16	1/16	1/2	3/4
13	15	r	1/8	5/16	r	2 3/8	2 3/8
Cisco	45	7 7/8	8 3/4	r	r	r	7/8
54 3/4	50	4 1/2	6	6 5/8	r	r	r
54 3/4	55	r	3 1/4	4	r	3	r
Citicp	10	s	1 3/4	1 7/8	s	r	5/16
11 1/2	12 1/2	1/16	7/16	1/2	1 1/16	r	1 5/16
11 1/2	15	r	s	1/8	3 3/4	s	3 5/8
Homstk	12 1/2	r	r	2 7/8	r	r	r
15 3/8	15	7/16	15/16	1 1/8	1/16	5/16	11/16
15 3/8	17 1/2	r	1/16	3/8	r	r	r
I B M	90	r	s	10 1/4	r	s	11/16
98 7/8	95	4	5 3/8	6 5/8	1/16	15/16	1 11/16
98 7/8	100	3/8	2 1/2	3 5/8	1 3/8	2 15/16	3 3/4
98 7/8	105	1/16	7/8	1 13/16	6 3/8	6 1/4	r
98 7/8	110	r	s	13/16	r	s	11 3/8
98 7/8	115	r	s	3/8	r	s	r
In Pap	65	8	s	r	r	s	r
74 3/8	70	3 1/4	3 3/4	r	r	3/4	1 7/8

Option & NY Close	Strike Price	Calls-Last Nov	Calls-Last Dec	Calls-Last Feb	Puts-Last Nov	Puts-Last Dec	Puts-Last Feb
17 5/8	20	r	r	r	3	r	r
Boeing	40	9	s	r	r	s	r
49 1/4	45	4 1/4	4 1/2	5 1/2	r	1/4	7/8
49 1/4	50	1/8	1 1/8	2 3/16	1	1 5/8	2 1/2
49 1/4	55	1/16	3/16	5/8	6	5 7/8	6 1/2
Bois C	22 1/2	r	r	r	r	r	1/2
25 1/2	25	9/16	r	1 5/8	1/16	r	r
C B S	150	5	r	r	r	1 7/8	r
155 1/2	155	1 9/16	4 3/8	10	1 1/8	r	7 3/8
155 1/2	160	1/4	2 7/8	7	4 1/2	7	r
Honwll	55	5 1/4	r	6 1/2	r	r	1 5/8
60	60	3/4	2 1/4	3 3/4	5/8	r	r
60	65	r	r	1 1/2	r	r	r
Humana	25	2 5/8	r	3 1/4	r	1/2	r
27 3/8	30	r	9/16	r	2 15/16	3	r
27 3/8	35	r	r	r	7 5/8	r	r
ImunRs	35	r	r	r	3/16	2	3 5/8
46	40	9	9 1/8	14 3/8	1/2	3 1/4	5 1/4
46	45	3 1/8	7	9 1/2	1 3/8	5 3/8	7
TelMex	35	r	r	11	r	r	5/16
45 1/4	40	5 1/4	5 3/4	7	r	3/8	r
45 1/4	45	11/16	2 1/4	3 7/8	r	1 13/16	2 15/16
45 1/4	50	r	5/8	1 13/16	r	5 1/4	6 1/8
Tribune	40	r	3/8	1 3/8	r	r	r
TritEn	40	8 1/2	8 3/4	11 1/8	r	7/8	r
48	45	3 1/2	5 1/2	7 3/8	1/4	2 1/2	r
48	50	9/16	3 1/2	5 3/4	2 3/16	r	r
48	55	r	1 5/8	4 1/4	r	r	r
UAL	20	r	s	r	1/16	s	r
131 7/8	25	6 3/4	s	r	r	s	5 1/4

Option & NY Close	Strike Price	Calls-Last Nov	Calls-Last Dec	Calls-Last Mar	Puts-Last Nov	Puts-Last Dec	Puts-Last Mar
Oracle	7 1/2	s	r	7 1/2	s	r	r
15 1/2	12 1/2	2 5/8	3	r	r	r	r
15 1/2	15	1/2	1 5/16	2 3/8	r	5/8	r
15 1/2	17 1/2	r	r	1 1/8	r	r	3
Pall	30	r	r	r	r	r	3/8
38 1/8	35	r	r	r	r	r	1
38 1/8	40	r	r	2	r	r	r
38 1/8	45	s	r	9/16	s	r	r
ParaCm	35	6 3/8	6 1/2	r	r	r	1/2
41 1/4	40	1 5/16	2 3/8	4	1/16	7/8	2 1/4

AMERICAN

Option & NY Close	Strike Price	Calls-Last Nov	Calls-Last Dec	Calls-Last Jan	Puts-Last Nov	Puts-Last Dec	Puts-Last Jan
Aetna	30	r	r	r	r	r	1/8
41 3/8	35	r	r	6 7/8	r	r	3/8
41 3/8	40	1 1/2	2 1/4	3	1/8	7/8	1 1/4
34 7/8	40	r	r	r	r	r	5 3/8
Cabltr	40	4 3/8	5 1/4	r	1/16	r	r
43 1/2	45	3/8	3 1/4	3 3/4	r	r	r
43 1/2	50	r	1 1/16	r	r	r	r
Cetus	12 1/2	s	s	6 1/2	s	s	r
19	15	3 7/8	r	4 1/2	r	r	r
19	17 1/2	1 1/2	2 1/16	2 5/8	r	3/8	3/4
19	20	1/16	1/2	1 3/16	r	r	r
Chips	7 1/2	r	r	1 1/8	r	r	r
8 3/8	10	r	r	5/16	r	r	r
Chiron	60	r	s	13 1/2	r	s	r

MOST ACTIVE OPTIONS

CHICAGO BOARD

	Sales	Last	Chg	N.Y. Close
CALLS				
SP100 Nov 370	50767	2 5/16	+3/16	371.20
SP100 Nov 365	30573	6 3/8	+1/2	371.20
SP100 Nov 375	9832	3/8	-1/16	371.20
SP100 Nov 360	9570	11 1/4	+3/4	371.20
SP100 Dec 370	8047	7 3/8	+3/4	371.20
SP100 Dec 375	7012	4 3/8	+1/2	371.20
SP100 Nov 380	5476	1/16	-1/16	371.20
I B M Nov 100	4699	3/8	-3/16	98 7/8
SP100 Dec 365	4243	10 5/8	+1/8	371.20
TelMex Feb 50	3172	1 13/16	-1/16	45 1/4
PUTS				
SP100 Nov 370	33791	1 1/16	-1 1/16	371.20
SP100 Nov 365	28524	5/16	-3/8	371.20
SP100 Nov 360	13976	1/8	-1/8	371.20
SP100 Dec 365	10083	3 5/8	-1/2	371.20
SP100 Dec 360	7759	2 1/2	-5/16	371.20
SP100 Nov 355	5404	1/16	-1/16	371.20
SP100 Nov 375	5266	4 1/8	-1 1/4	371.20
SP100 Dec 370	4556	5 1/4	-1/2	371.20
SP100 Dec 340	3482	5/8	-1/16	371.20
SP500 Nov 390	3445	1/4	-3/16	397.41

AMERICAN

	Sales	Last	Chg	N.Y. Close
CALLS				
Glaxo Dec 30	3534	1 11/16	+1 11/16	30 3/8
RJR Nb Nov 7 1/2	2964	2 5/8	-1/16	10 1/8
MMIdx Nov 325	2496	3 1/8	+1	327.25
Dig Eq Nov 60	2214	2 7/8	+5/8	62 3/4
Ph Mor Nov 70	1946	3/4	-1/8	70 5/8
RJR Nb Dec 7 1/2	1900	2 5/8	-1/16	10 1/8
TelMex Feb 50	1821	1 7/8	+3/16	45 1/4
MMIdx Nov 320	1632	7 1/2	+1 1/4	327.25
Gillet Dec 45	1599	1 3/8	+1/8	44 3/4
TritEn Nov 50	1549	3/4	+9/16	48
PUTS				
Disney Nov 110	2089	1	+5/8	110 3/8
RJR Nb Dec 10	2010	1/4	...	10 1/8
RJR Nb Dec 7 1/2	1900	1/16	...	10 1/8
RJR Nb Nov 7 1/2	1900	1/16	...	10 1/8
MMIdx Nov 325	1627	11/16	-13/16	327.25
Disney Dec 110	1054	3 1/4	+1	110 3/8
MMIdx Nov 320	1035	1/8	-1/4	327.25
Chase Nov 15	1000	1/8	...	18 1/4
Ph Mor Nov 70	925	3/16	...	70 5/8
Pfizer Nov 70	779	11/16	+5/16	70 1/8

PHILADELPHIA

	Sales	Last	Chg	N.Y. Close
CALLS				
NCNB Dec 35	2225	4 1/8	+1/4	39
NCNB Nov 35	2130	4	+1/8	39
RJR Nb Nov 10	1800	3/16	-1/16	10 1/8
RJR Nb Dec 10	1750	7/16	...	10 1/8
RJR Nb Nov 7 1/2	1049	2 5/8	+1/16	10 1/8
Synrgn Nov 65	965	3/8	-2 1/2	59 7/8
HomeD Nov 60	805	1	-3/8	60 3/4
HomeD Dec 60	775	3 1/8	-1/8	60 3/4
Synrgn Nov 60	619	1 13/16	-4 5/16	59 7/8
Abbt L Dec 55	571	6 3/8	-3/8	61 1/8
PUTS				
HomeD Dec 55	2187	3/4	+1/4	60 3/4
Comeric Jan 50	2000	1	-1 3/8	52 1/4
Synrgn Nov 65	1074	5 1/2	+3 5/8	59 7/8
BkBost Nov 12 1/2	1020	1/4	...	12 1/2
SecPac Dec 35	578	4 3/8	+1/8	31 3/8
RJR Nb Nov 12 1/2	515	2 3/8	...	10 1/8
HomeD Dec 50	505	1/4	+1/16	60 3/4
RJR Nb Dec 10	500	1/4	...	10 1/8
Abbt L Feb 60	434	2	-3/16	61 1/8
HomeD Nov 60	426	1/4	...	60 3/4

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Non-Dividend-Paying Stocks

To derive the lower price bounds and the put-call parity relations for options on non-dividend-paying stocks, simply set the cost-of-carry rate, b, equal to the riskless rate of interest, r, in the relations presented in Chapter 10. The only cost of carrying the stock is interest.

The lower price bounds for the European call and put options are

$$c(S,T;X) \geq \max[0, S - Xe^{-rT}] \tag{13.1a}$$

and

$$p(S,T;X) \geq \max[0, Xe^{-rT} - S], \tag{13.1b}$$

respectively, and the lower price bounds for the American call and put options are

$$C(S,T;X) \geq \max[0, S - Xe^{-rT}] \tag{13.2a}$$

and

$$P(S,T;X) \geq \max[0, X - S], \tag{13.2b}$$

respectively. The put-call parity relation for non-dividend-paying European stock options[3] is

$$c(S,T;X) - p(S,T;X) = S - Xe^{-rT}, \tag{13.3a}$$

and the put-call parity relation for American options on non-dividend-paying stocks is

$$S - X \leq C(S,T;X) - P(S,T;X) \leq S - Xe^{-rT}. \tag{13.3b}$$

For non-dividend-paying stock options, the American call option will not rationally be exercised early, while the American put option may be.[4]

Dividend-Paying Stocks

If dividends are paid during the option's life, the above relations must reflect the stock's drop in value when the dividends are paid. To manage this modification, we assume that the underlying stock pays a single dividend during the option's life

[3]The original formulation of put-call parity for European stock options is contained in Stoll (1969).
[4]For proofs of any of the relations (13.1a) through (13.3b), see Chapter 10.

at a time that is known with certainty. The dividend amount is D and the time to ex-dividend is t.[5]

If the amount and the timing of the dividend payment is known, the lower price bound for the European call option on a stock is

$$c(S, T; X) \geq \max[0, S - De^{-rt} - Xe^{-rT}]. \tag{13.4a}$$

In this relation, the current stock price is reduced by the present value of the promised dividend. Because a European-style option cannot be exercised before maturity, the call option holder has no opportunity to exercise the option while the stock is selling cum dividend. In other words, to the call option holder, the current value of the underlying stock is its observed market price less the amount that the promised dividend contributes to the current stock value, that is, $S - De^{-rt}$. To prove this pricing relation, we use the same arbitrage transactions as in Chapter 10, except we use the reduced stock price $S - De^{-rt}$ in place of S.

The lower price bound for the European put option on a stock is

$$p(S, T; X) \geq \max[0, Xe^{-rT} - S + De^{-rt}]. \tag{13.4b}$$

Again, the stock price is reduced by the present value of the promised dividend. Unlike the call option case, however, this serves to increase the lower price bound of the European put option. Because the put option is the right to sell the underlying stock at a fixed price, a discrete drop in the stock price such as that induced by the payment of a dividend serves to increase the value of the option. An arbitrage proof of this relation is straightforward when the stock price, net of the present value of the dividend is used in place of the commodity price.

The lower price bounds for American stock options are slightly more complex. In the case of the American call option, for example, it may be optimal to exercise just prior to the dividend payment because the stock price falls by an amount D when the dividend is paid. The lower price bound of an American call option expiring at the ex-dividend instant would be 0 or $S - Xe^{-rt}$, whichever is greater. On the other hand, it may be optimal to wait until the call option's expiration to exercise. The lower price bound for a call option expiring normally is (13.4a).[6] Combining the two results, we get

$$C(S, T; X) \geq \max[0, S - Xe^{-rt}, S - De^{-rt} - Xe^{-rT}]. \tag{13.5a}$$

[5] Generalizations of the results to cases where there are more than one known dividend are derived in the same manner as the single dividend results shown here.

[6] Recall that in Chapter 10 we showed that an American call is never optimally exercised early if $b \geq r$. Between dividends, the cost-of-carry rate is r, so exercise is not optimal. At the ex-dividend instant, however, the call option holder may wish to exercise to capture the dividend.

The last two terms on the right-hand side of (13.5a) provide important guidance in deciding whether to exercise the American call option early, just prior to the ex-dividend instant. The second term in the squared brackets is the present value of the early exercise proceeds of the call. If the amount is less than the lower price bound of the call that expires normally, that is, if

$$S - Xe^{-rt} < S - De^{-rt} - Xe^{-rT}, \tag{13.6}$$

the American call option will not be exercised just prior to the ex-dividend instant. To see why, simply rewrite (13.6) so it reads

$$D < X[1 - e^{-r(T-t)}]. \tag{13.7}$$

In other words, the American call will not be exercised early if the dividend captured by exercising prior to the ex-dividend date is less than the interest implicitly earned by deferring exercise until expiration.

Figure 13.1 depicts a case in which early exercise could occur at the ex-dividend instant, t. Just prior to ex-dividend, the call option may be exercised yielding proceeds $S_t + D - X$, where S_t is the ex-dividend stock price. An instant later, the option is left unexercised with value $c(S_t, T - t; X)$, where $c(\cdot)$ is the European call option formula. Thus, if the ex-dividend stock price, S_t, is above the critical ex-dividend stock price where the two functions intersect, S_t^*, the option holder will choose to exercise her option early just prior to the ex-dividend instant. On the other hand, if $S_t \leq S_t^*$, the option holder will choose to leave her position open until the option's expiration.

FIGURE 13.1 American call option price as a function of the ex-dividend stock price immediately prior to the ex-dividend instant. Early exercise may be optimal.

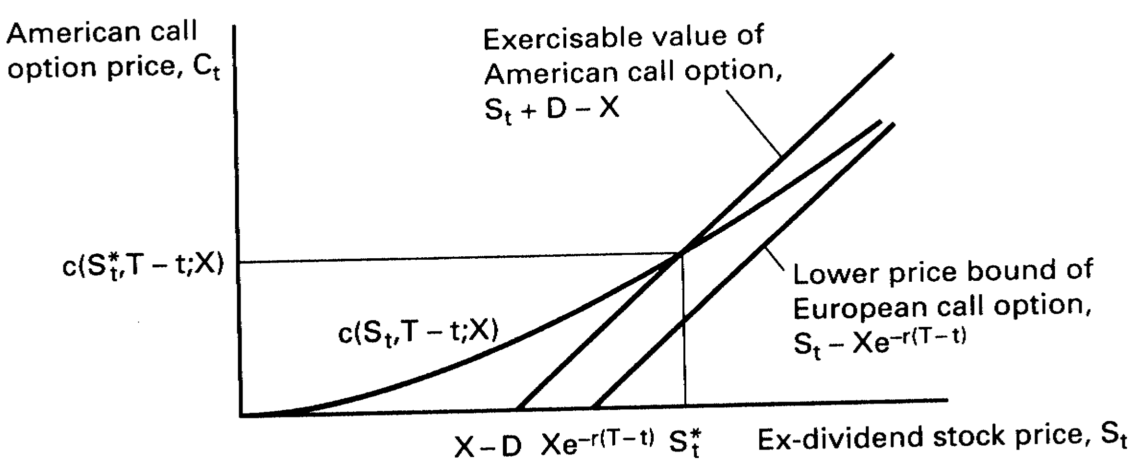

FIGURE 13.2 American call option price as a function of the ex-dividend stock price immediately prior to the ex-dividend instant. Early exercise will not be optimal.

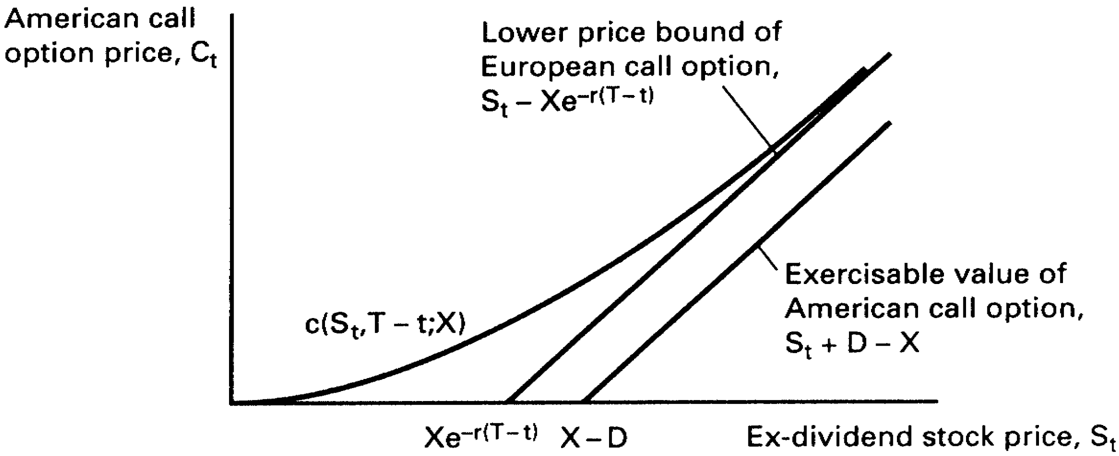

Figure 13.2 depicts a case in which early exercise will not occur at the ex-dividend instant, t. Early exercise will not occur if the functions, $S_t + D - X$ and $c(S_t, T - t; X)$ do not intersect, as is depicted in Figure 13.2. In this case, the lower boundary condition of the European call, $S_t - Xe^{-r(T-t)}$, lies above the early exercise proceeds, $S_t + D - X$, and hence the call option will not be exercised early. Stated explicitly, early exercise is not rational if

$$S_t + D - X < S_t - Xe^{-r(T-t)}.$$

This condition for no early exercise is the same as (13.6), where S_t is the ex-dividend stock price and where the investor is standing at the ex-dividend instant, t. The condition can also be written as

$$D < X[1 - e^{-r(T-t)}]. \quad \textbf{(13.7)}$$

In words, if the ex-dividend stock price decline—the dividend—is less than the present value of the interest income that would be earned by deferring exercise until expiration, early exercise will not occur. When condition (13.7) is met, the value of the American call is simply the value of the corresponding European call.

The lower price bound of an American put option is somewhat different. In the absence of a dividend, an American put may be exercised early. In the presence of a dividend payment, however, there is a period just prior to the ex-dividend date when early exercise is suboptimal. In that period, the interest earned on the exercise proceeds of the option is less than the drop in the stock price from the payment of the dividend. If t_n represents a time prior to the dividend payment at time t, early exercise is suboptimal, where $(X - S)e^{r(t-t_n)}$ is less than $(X - S + D)$. Rearranging, early exercise will not occur between t_n and t if[7]

$$t_n > t - \frac{\ln(1 + \frac{D}{X-S})}{r}. \tag{13.8}$$

Early exercise will become a possibility again immediately after the dividend is paid. Overall, the lower price bound of the American put option is

$$P(S,T;X) \geq \max[0, X - (S - De^{-rt})]. \tag{13.5b}$$

Put-call parity for European options on dividend-paying stocks also reflects the fact that the current stock price is deflated by the present value of the promised dividend, that is,

$$c(S,T;X) - p(S,T;X) = S - De^{-rt} - Xe^{-rT}. \tag{13.9}$$

That the presence of the dividend reduces the value of the call and increases the value of the put is again reflected here by the fact that the term on the right-hand side of (13.9) is smaller than it would be if the stock paid no dividend.

Put-call parity for American options on dividend-paying stocks is represented by a pair of inequalities, that is,

$$S - De^{-rt} - X \leq C(S,T;X) - P(S,T;X) \leq S - De^{-rt} - Xe^{-rT}. \tag{13.10}$$

To prove the put-call parity relation (13.10), we consider each inequality in turn. The left-hand side condition of (13.10) can be derived by considering the values of a portfolio that consists of buying a call, selling a put, selling the stock, and lending $X + De^{-rt}$ risklessly. Table 13.2 contains these portfolio values.

In Table 13.2, it can be seen that, if all of the security positions stay open until expiration, the terminal value of the portfolio will be positive, independent of whether the terminal stock price is above or below the exercise price of the options. If the terminal stock price is above the exercise price, the call option is exercised, and the stock acquired at exercise price X is used to deliver, in part, against the short stock position. If the terminal stock price is below the exercise price, the put

[7] It is possible that the dividend payment is so large that early exercise prior to the dividend payment is completely precluded. For example, consider the case where $X = 50$, $S = 40$, $D = 1$, $t = .25$ and $r = .10$. Early exercise is precluded if $t_n = .25 - \ln[1 - 1/(50 - 40)]/.10 = -.7031$. Because the value is negative, the implication is that there is no time during the current dividend period (i.e., from 0 to t) where it will not pay the American put option holder to wait until the dividend is paid to exercise his option.

TABLE 13.2 Arbitrage transactions for establishing put-call parity for American stock options. $S - De^{-rt} - X \leq C(S,T;X) - P(S,T;X)$

Position	Initial Value	Ex-Dividend Day (t)	Put Exercised Early (τ): Intermediate Value	Put Exercised Normally (T): Terminal Value $\tilde{S}_T \leq X$	Put Exercised Normally (T): Terminal Value $\tilde{S}_T > X$
Buy American Call	$-C$		$\tilde{C}_\tau$	0	$\tilde{S}_T - X$
Sell American Put	P		$-(X - \tilde{S}_\tau)$	$-(X - \tilde{S}_T)$	0
Sell Stock	S	$-D$	$-\tilde{S}_\tau$	$-\tilde{S}_T$	$-\tilde{S}_T$
Lend De^{-rt}	$-De^{-rt}$	D			
Lend X	$-X$		$Xe^{r\tau}$	Xe^{rT}	Xe^{rT}
Net Portfolio Value	$-C + P + S - De^{-rt} - X$	0	$\tilde{C}_\tau + X(e^{r\tau} - 1)$	$X(e^{rT} - 1)$	$X(e^{rT} - 1)$

is exercised. The stock received in the exercise of the put is used to cover the short stock position established at the outset. In the event the put is exercised early at time τ, the investment in the riskless bonds is more than sufficient to cover the payment of the exercise price to the put option holder, and the stock received from the exercise of the put is used to cover the stock sold when the portfolio was formed. In addition, an open call option position that may still have value remains. In other words, by forming the portfolio of securities in the proportions noted above, we have formed a portfolio that will never have a negative future value. If the future value is certain to be nonnegative, the initial value must be nonpositive, or the left-hand inequality of (13.10) holds.

The right-hand side of (13.10) may be derived by considering the portfolio used to prove European put-call parity. Table 13.3 contains the arbitrage portfolio transactions. In this case, the terminal value of the portfolio is certain to equal zero, should the option positions stay open until that time. In the event the American call option holder decides to exercise the call option early, the portfolio holder uses his long stock position to cover his stock obligation on the exercised call and uses the exercise proceeds to retire his outstanding debt. After these actions are taken, the portfolio holder still has an open long put position and cash in the amount of $X[1 - e^{-r(T-t)}]$. Since the portfolio is certain to have nonnegative outcomes, the initial value must be nonpositive or the right-hand inequality of (13.10) must hold.

13.3 VALUATION OF CALL OPTIONS ON STOCKS

European Call Option on Non-Dividend-Paying Stocks

A non-dividend-paying stock has a cost-of-carry rate, b, equal to the riskless rate of interest, r, so, using equation (11.25) from Chapter 11, the *valuation equation*

TABLE 13.3 Arbitrage transactions for establishing put-call parity for American stock options. $C(S,T;X) - P(S,T;X) \leq S - De^{-rt} - Xe^{-rT}$

Position	Initial Ex-Dividend	Ex-Dividend Day (t) Value	Call Exercised Early (τ) Intermediate Value	Call Exercised Normally (T) Terminal Value $\tilde{S}_T \leq X$	$\tilde{S}_T > X$
Sell American Call	C		$-(\tilde{S}_\tau - X)$	0	$-(\tilde{S}_T - X)$
Buy American Put	$-P$		$\tilde{P}_\tau$	$X - \tilde{S}_T$	0
Buy Stock	$-S$	D	$\tilde{S}_\tau$	$\tilde{S}_T$	$\tilde{S}_T$
Borrow De^{-rt}	De^{-rt}	$-D$			
Borrow Xe^{-rT}	Xe^{-rT}		$-Xe^{-r(T-\tau)}$	$-X$	$-X$
Net Portfolio Value	$C - P - S$ $+De^{-rt} + Xe^{-rT}$	0	$\tilde{P}_\tau$ $+X[1 - e^{-r(T-t)}]$	0	0

of a European call option on a non-dividend-paying stock[8] is

$$c(S,T;X) = SN(d_1) - Xe^{-rT}N(d_2), \tag{13.11}$$

where

$$d_1 = \frac{\ln(S/X) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

European Call Option on Dividend-Paying Stocks

In the presence of a known discrete dividend, option valuation remains relatively straightforward. In place of assuming that future stock prices are lognormally distributed, we now assume that stock prices, net of the present value of the escrowed dividend, are lognormally distributed. The dividend to be paid during the option's life is a known amount on a known date. At the ex-dividend instant, the stock price drops by the amount of the dividend. The stock price net of the present value of the dividend, however, remains unchanged.

Under this modified assumption, the valuation equations for European options maintain the same structure as in Chapter 11. The only change is that the current stock price net of the present value of the promised dividend,

$$S^x = S - De^{-rt}, \tag{13.12}$$

[8]This equation is often referred to as simply the *Black–Scholes formula*, given the important impact that the Black–Scholes (1973) paper has had on the theory of option pricing and, more generally, the practice of finance.

is substituted for the stock price parameter in the European call option pricing formula (13.11). The *valuation equation of a European call option on a dividend-paying stock* is

$$c(S,T;X) = S^x N(d_1) - Xe^{-rT}N(d_2), \tag{13.13}$$

where

$$d_1 = \frac{\ln(S^x/X) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Naturally, the value of the call decreases as a result of the cash disbursement on the stock.

American Call Option on Non-Dividend-Paying Stocks

In Chapter 11, we showed that if the cost-of-carry rate of the underlying commodity is greater than or equal to the riskless rate of interest (i.e., $b \geq r$), the American call option on the commodity will never be optimally exercised early and its value equals the value of a European call option. In the case of a non-dividend-paying stock, the cost-of-carry rate equals the riskless rate of interest. Hence, the *valuation equation of a American call option on a non-dividend-paying stock* is (13.11).

American Call Option on Dividend-Paying Stocks

When a stock pays a dividend, the problem of valuing an American call is more complex. Whereas an American call option on a non-dividend-paying stock will never optimally be exercised prior to expiration, an American call option on a dividend-paying stock *may* be. This situation arises because the stock price falls at the ex-dividend instant, which causes the call to drop in value. It may be optimal for the American call option holder to exercise his option just prior to this ex-dividend stock price drop.

The easiest way to derive the valuation equation for the American call option on a dividend-paying stock is to use the risk-neutral option valuation. Consider the American call option holder's dilemma as depicted in Figure 13.1. At time t, the American option holder will exercise his option if $S_t > S_t^*$ and will leave his position open if $S_t \leq S_t^*$. And, if he leaves his position open at t, his option will have terminal values $\tilde{S}_T - X$ if $S_T > X$ and 0 if $S_T \leq X$. Recognizing that the option's payoffs will occur at one of two points in time (i.e., just prior to the early exercise instant or at expiration) allows us to write the current value of the call as the present value of the expected future payoffs, that is,

$$\begin{aligned} C(S,T;X) &= e^{-rt}E(\tilde{C}_t|S_t > S_t^*) + e^{-rT}E(\tilde{C}_T|S_t \leq S_t^*) \\ &= e^{-rt}E(\tilde{S}_t + D - X|S_t > S_t^*) \\ &\quad + e^{-rT}E(\tilde{S}_T - X|S_t \leq S_t^* \text{ and } S_T > X) \end{aligned}$$

$$= e^{-rt}[E(\tilde{S}_t|S_t > S_t^*) - (X - D)\text{Prob}(S_t > S_t^*)]$$
$$+ e^{-rT}[E(\tilde{S}_T|S_t \leq S_t^* \text{ and } S_T > X)$$
$$- X\text{Prob}(S_t \leq S_t^* \text{ and } S_T > X)]. \quad \textbf{(13.14)}$$

Assuming that the future stock price net of the present value of the promised dividend is lognormally distributed, the expected values on the right-hand side of (13.14) become the *valuation equation of an American call option on a dividend-paying stock*:

$$C(S, T; X) = e^{-rt}[S^x e^{rt} N_1(b_1) - (X - D)N_1(b_2)]$$
$$+ e^{-rT}[S^x e^{rT} N_2(a_1, -b_1; -\sqrt{t/T})$$
$$- XN_2(a_2, -b_2; -\sqrt{t/T})] \quad \textbf{(13.15a)}$$

$$= S^x[N_1(b_1) + N_2(a_1, -b_1; -\sqrt{t/T})]$$
$$- Xe^{-rT}[N_1(b_2)e^{r(T-t)} + N_2(a_2, -b_2; -\sqrt{t/T})]$$
$$+ De^{-rt}N_1(b_2), \quad \textbf{(13.15b)}$$

where

$$a_1 = \frac{\ln(S^x/X) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}, \quad a_2 = a_1 - \sigma\sqrt{T},$$

$$b_1 = \frac{\ln(S^x/S_t^*) + (r + .5\sigma^2)t}{\sigma\sqrt{t}}, \quad b_2 = b_1 - \sigma\sqrt{t},$$

$N_1(b)$ is the cumulative univariate normal density function with upper integral limit b[9] and $N_2(a, b; \rho)$ is the cumulative bivariate normal density function with upper integral limits, a and b, and correlation coefficient, ρ.[10] S_t^* is the ex-dividend stock price for which

$$c(S_t^*, T - t; X) = S_t^* + D - X, \quad \textbf{(13.15c)}$$

as noted earlier in the discussion of Figure 13.1.[11]

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[9]Recall that the function $N(b)$ was used for the first time in Chapter 11. The subscript 1 is used here only to contrast the univariate integral from the bivariate integral.

[10]More details about the bivariate normal probability, as well as a method of computation and a numerical example, are contained in Appendix 13.1.

[11]Roll (1977) provides a framework for analytically valuing the American call option. The valuation formula (13.15b) is from Whaley (1981).

In equation (13.15a), the American call formula is the sum of the present values of two conditional expected values—the present value of the expected early exercise value of the option conditional on early exercise, $S^x N_1(b_1) - (X - D)e^{-rt}N_1(b_2)$, and the present value of the expected terminal exercise value of the call conditional on no early exercise, $S^x N_1(a_1, -b_1; -\sqrt{t/T}) - Xe^{-rT}N_2(a_2, -b_2; -\sqrt{t/T})$. The term $N_1(b_2)$ is the probability that the call option will be exercised early, and the term $N_2(a_2, -b_2; -\sqrt{t/T})$ is the joint probability that the call option will not be exercised early and will be in-the-money at expiration.

Note that as the amount of the dividend approaches the present value of the interest income that would be earned by deferring exercise until expiration, that is, $D \to X[1 - e^{-r(T-t)}]$, the value of S_t^* approaches $+\infty$, the values of $N_1(b_1)$ and $N_1(b_2)$ approach 0, the values of $N_2(a_1, -b_1; -\sqrt{t/T})$ and $N_2(a_2, -b_2; -\sqrt{t/T})$ approach $N_1(a_1)$ and $N_1(a_2)$, respectively, and the American call option formula (13.15b) becomes the dividend-adjusted Black–Scholes European call option formula (13.13).

EXAMPLE 13.1

Compute the value of an American-style call option whose exercise price is \$50 and whose time to expiration is 90 days. Assume the riskless rate of interest is 10 percent annually, the underlying stock price is \$50, the standard deviation of the rate of return of the stock is 30 percent, and the stock pays a dividend of \$2 in exactly 60 days.

We compute the value of this call first using the European formula (13.13) and then using the American formula (13.15b). In this way, we can identify the value of the early exercise premium on the call.

The current stock price net of the present value of the promised dividend is

$$S^x = 50 - 2e^{-.10(60/365)} = 48.033,$$

so the European call value can be computed as

$$c = 48.033N_1(d_1) - 50e^{-.10(90/365)}N_1(d_2),$$

where

$$d_1 = \frac{[\ln(48.033/50) + (.10 + .5(.30)^2)(90/365)]}{.30\sqrt{90/365}} = -.029$$

and $d_2 = -.029 - .149 = -.178$. The probabilities $N_1(-.029)$ and $N_1(-.178)$ are .488 and .429, so the European call value is

$$c = 48.033(.488) - 48.782(.429) = 2.51.$$

Prior to applying the valuation equation for the American call option on a dividend-paying stock (13.15b), we must determine if there is a chance of early exercise. Recall that if condition (13.7) holds, the call will not be exercised early. Substituting the exercise values into (13.7), we find

$$2 > 50[1 - e^{-.10(90-60)/365}] = .409,$$

showing that early exercise is not precluded and that formula (13.15b) should be used.

The value of the American call is now computed as

$$\begin{aligned} C = {} & 48.033[N_1(b_1) + N_2(a_1, -b_1; -\sqrt{t/T})] \\ & - 50e^{-.10(90/365)}[N_1(b_2)e^{.10(30/365)} + N_2(a_2, -b_2; -\sqrt{t/T})] \\ & + 2e^{-.10(60/365)}N_1(b_2), \end{aligned}$$

where $t/T = (60/365)/(90/365) = 60/90 = .667$,

$$a_1 = \frac{\ln(48.033/50) + (.10 + .5(.30)^2)(90/365)}{.30\sqrt{90/365}} = -.029$$

$$a_2 = -.029 - .30\sqrt{90/365} = -.178$$

$$b_1 = \frac{\ln(48.033/49.824) + (.10 + .5(.30)^2)(60/365)]}{.30\sqrt{60/365}} = -.105$$

and

$$b_2 = -.105 - .30\sqrt{60/365} = -.227.$$

The values b_1 and b_2 depend on the critical ex-dividend stock price S_t^*, which is determined by

$$c(S_t^*, 30/365; 50) = S_t^* + 2 - 50$$

and, in this example, equals 49.824. The bivariate normal probabilities are $N_2(a_1, -b_1; -\rho) = .1135$ and $N_2(a_2, -b_2; -\rho) = .1056$, and the univariate normal probabilities are $N_1(b_1) = .4581$ and $N_1(b_2) = .4103$. The value of the American call is 2.931; hence, the early exercise premium on the American option is 2.931 − 2.513 = .418.

TABLE 13.4 Simulated American and European call option values on a stock with a known discrete dividend. The call option has an exercise price of $50 and a time to expiration of 90 days. The riskless rate of interest is 10 percent, and the standard deviation of the stock return is 30 percent. The stock pays a dividend of $2 in 60 days.

Stock Price S	European Call Price $c(S,T;X)$	American Call Price $C(S,T;X)$	Early Exercise Premium ϵ_C
40	.126	.136	.011
45	.760	.867	.107
50	2.515	2.931	.418
55	5.610	6.481	.871
60	9.726	10.974	1.248

The size of the early exercise premium of an American call option on a dividend-paying stock becomes larger as the option goes deeper in the money. In Table 13.4, we extend the results of Example 13.1 by allowing the stock price to vary from $40 to $60. It is interesting to note that the dividend payment induces a fairly large early exercise premium on the call option, particularly when the call is deep in-the-money. At a $60 stock price, for example, the value of the early exercise premium is about $1.25, more than 11 percent of the call's overall value.

Dividend Spreads

In practice, not all call options are exercised when they should be. And, when they are not, there are ways to profit risklessly. Consider, for example, two in-the-money call options written on a stock that is about to pay a dividend. Assume the deeper in-the-money call is sold and the other is purchased. Now, on the day before ex-dividend, exercise the purchased option and wait. If the holder of the deeper in-the-money call exercises her option before ex-dividend, deliver the stock received from the exercise of the purchased option and pay the net difference between the exercise prices of the options. On the other hand, suppose the holder of the deeper in-the-money call option forgets to exercise her option. In this case, sell the stock the next morning and buy back the remaining option. In the first case, profit equals zero, and, in the second, a profit in the amount of the dividend would be received. A numerical example may serve to clarify this strategy.

Assume that a stock is currently priced at $60 and will pay a $2.00 dividend tomorrow. Call options with exercise prices of 50 and 55 and time to expiration of 30 days are priced at $10.01 and $5.01, respectively. (The riskless rate of interest is assumed to be 10 percent, and the standard deviation of stock returns is 30 percent.) Now, assume the 50 call is sold, and the 55 call is purchased, yielding net proceeds at the outset of $5.00. At the end of the day, the investor exercises the 55 call, receiving proceeds S_{t-1} + 2.00 − 55. (Day $t - 1$ is the day before ex-dividend, and the notation S_{t-1} is the stock price net of the value of the escrowed dividend.) If the 50 call option is exercised against the investor before ex-dividend (which will not be known until the next day before market opening), the investor's obligation is $-(S_t + 2.00 - 50)$, and the net terminal value is $S_t + 2.00 - 55 - S_t - 2.00 + 50$ or −$5, exactly a wash, considering $5 was collected up front. However, if the 50 call option is not exercised, the investor goes into the next morning with a long position in the stock (acquired from exercising the 55 call) and a short position in the 50 call, which has a value $S_t + 2.00 - 55 - C_t$. For all intents and purposes, this position is riskless because the uncertainty in the value of the long stock position is offset by that of the short in-the-money call. The outstanding call, being deep in-the-money, is selling for about its floor value of $S_t - 50$, so the net value of the position is 2.00 − 55 + 50 or −$3. Net of the initial cash receipt of $5, the profit is $2.00, exactly the amount of the dividend. This strategy is usually called a *dividend capture* or a *dividend spread.*

13.4 VALUATION OF PUT OPTIONS ON STOCKS

European Put Option on Non-Dividend-Paying Stocks

The *valuation equation of a European put option on a non-dividend-paying stock* is

$$p(S,T;X) = Xe^{-rT}N(-d_2) - SN(d_1), \tag{13.16}$$

where

$$d_1 = \frac{\ln(S/X) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

This result follows straightforwardly from substituting the riskless rate of interest, r, for the cost-of-carry rate, b, in equation (11.28) from Chapter 11.

European Put Option on Dividend-Paying Stocks

Like the European call option on a dividend-paying stock, the European put option on a dividend-paying stock is obtained by substituting the stock price net of the present value of the dividend, $S^x = S - De^{-rt}$, for the stock price parameter in the European option pricing formula. The *valuation equation of a European put option on a dividend-paying stock* is

$$p(S,T;X) = Xe^{-rT}N(-d_2) - S^xN(-d_1), \tag{13.17}$$

where

$$d_1 = \frac{\ln(S^x/X) + (r + .5\sigma^2)T}{\sigma\sqrt{T}}, \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Note that the put value increases as a result of the cash disbursement on the stock.

American Put Option on Non-Dividend-Paying Stocks

The American put option on a non-dividend-paying stock is not a tractable problem from a mathematical standpoint, as was noted in Chapter 11. For this reason, numerical methods must be used to approximate the value of the put. The approach used here is the binomial method.[12] The binomial method assumes the stock price moves in discrete jumps over discrete intervals of time. The up-and-down steps in stock price are defined as a proportion of the stock price at the previous interval. If the current stock price is S_0, the stock price at the end of the first interval is either uS_0 or dS_0. If the total number of time steps is defined as n, where $\Delta t = T/n$ and T is the time to expiration of the option, there are $n + 1$ possible stock prices at the option's expiration. This binomial lattice is illustrated in Figure 13.3. The length of each interval or time step in the figure is Δt. The factors u and d are defined as[13]

$$u = e^{\sigma\sqrt{\Delta t}} \tag{13.18a}$$

and

$$d = \frac{1}{u}. \tag{13.18b}$$

The risk-neutral probabilities of up and down movements are

$$p = \frac{r^* - d}{u - d} \tag{13.18c}$$

and $1 - p$, respectively, where $r^* = e^{r\Delta t}$.

Figure 13.4 provides a numerical illustration of the binomial lattice for the stock price. The current stock price, S_0, is 50, the riskless rate of interest, r, is 10 percent, and the standard deviation of stock returns, σ, is 30 percent. The time to expiration of the option, T, is 90 days, and the number of time steps, n, is 90. The

[12] See Cox, Ross, and Rubinstein (1979).

[13] The factors are consistent with the Black–Scholes model. See Cox, Ross, and Rubinstein (1979) for details.

FIGURE 13.3 Possible paths that the stock price may follow under the binomial model.

Stock price at end of time interval:

0	1	2	3	4	⋯	n (even)	or	n (odd)
						u^nS_0		u^nS_0
				u^4S_0	⋯			
			u^3S_0			⋮		⋮
		u^2S_0		u^2S_0	⋯			
	uS_0		uS_0					uS_0
S_0		S_0		S_0	⋯	S_0		
	dS_0		dS_0					dS_0
		d^2S_0		d^2S_0	⋯	⋮		
			d^3S_0					⋮
				d^4S_0	⋯			
						d^nS_0		d^nS_0

FIGURE 13.4 Possible paths that the stock price may follow under the binomial model, where the current stock price is 50, the riskless rate of interest is 10 percent annually, and the standard deviation of stock returns is 30 percent annually. The time to expiration of the option is 90 days, and the number of time steps is also 90. The time increment Δt is, therefore, 1 day or .00274 years.

Stock price at end of time interval:

0	1	2	3	4	⋯	90 days
						205.46
				53.24	⋯	
			52.41			⋮
		51.60		51.60	⋯	
	50.79		50.79			
50		50		50	⋯	50
	49.22		49.22			
		48.45		48.45	⋯	⋮
			47.70			
				46.96	⋯	
						12.17

size of the time increment Δt is, therefore, one day or .00274 years. Based on this information, the factors u and d are

$$u = e^{.30\sqrt{.00274}} = 1.01583$$

and

$$d = \frac{1}{1.01583} = .98442.$$

Also, the probabilities of up-and-down movements are

$$p = \frac{e^{.10(.00274)} - .98442}{1.01583 - .98442} = .5048$$

and $1 - p = .4952$. Note that, in Figure 13.4, possible stock prices range from $S_0 d^n = 50(.98442)^{90} = 12.17$ to $S_0 u^n = 50(1.0158267)^{90} = 205.46$ at the option's expiration.

With the stock price lattice computed, the approximation method starts at the end of the option's life and works back to the present, one increment, Δt, at a time. At the end of the option's life (column n in the figure), the option value at each stock price node is given by the intrinsic value of the put option, that is,

$$P_{n,j}(S_{n,j}) = \begin{cases} 0 & \text{where} \quad S_{n,j} > X \\ X - S_{n,j} & \text{where} \quad S_{n,j} \leq X. \end{cases} \tag{13.19}$$

The option values one step back in time (in column $n - 1$) are computed by taking the present value of the expected future value of the option. At any point j in column $n - 1$, the stock price can move up with probability p or down with probability $1 - p$. The value of the option at time n if the stock price moves up is $P_{n,j+1}$ and if the stock price moves down is $P_{n,j}$. The present value of the expected future value of the option is, therefore,

$$P_{n-1,j} = \frac{pP_{n,j+1} + (1-p)P_{n,j}}{r^*}, \tag{13.20}$$

where $r^* = e^{r\Delta t}$. Using this present value formulation, all of the option values in column $n - 1$ may be identified.

Before proceeding back another time increment, Δt, in the valuation, it is necessary to see if any of the computed option values are below their early exercise proceeds at the respective nodes, $X - S_{n-1,j}$. If the exercise proceeds are greater than the computed option value, the computed value is replaced with the early exercise proceeds. If they are not, the value is left undisturbed. If this step is not performed, the procedure will produce the value of a European put option.

Once the checks are performed, we go to column $n - 2$, repeat the steps and so on back through time. Eventually, we will work our way back to time 0, and the current value of the American put option (in column 0) will be identified.

To complete the binomial method illustration, suppose that the stock price lattice shown in Figure 13.4 underlies a 90-day American put option with an exercise price of 50. Applying the binomial method, the value of the American put is \$2.475. The value of this put using the European formula (13.16) is \$2.364, which means that the early exercise premium of the American put is worth about 11.1¢. Note that the value produced by the binomial method for the European put (by not checking for the early exercise constraints) is \$2.355, which is different from the \$2.364 obtained using the European formula. This is error due to the fact that the binomial method is only an approximation. In general, the accuracy of the binomial method increases with the number of time steps, holding other factors constant.

American Put Option on Dividend-Paying Stocks

The binomial method is also well suited to handle the case of valuing an American put option on a dividend-paying stock. If the dividends paid during the put's life are known with certainty, we first subtract the present value of the dividends from the current stock price, that is,

$$S_0^x = S_0 - \sum_{i=1}^{n} D_i e^{-rt_i}, \tag{13.21}$$

where D_i (t_i) is the amount of (time to) the i-th dividend paid during the option's life and S_0 is the current stock price. Next, we set up the binomial lattice, beginning with S_0^x rather than S_0. That is, if the current stock price net of dividends is S_0^x, the stock price at the end of the first time interval is either uS_0^x or dS_0^x. The values of u, d, and p are computed as (13.18a), (13.18b), and (13.18c).

With the stock price lattice (net of dividends) computed, the approximation method starts at the end of the option's life and works back to the present. At the end of the option's life, the option value at each stock price node is given by the intrinsic value of the put (13.19), where S^x replaces S. The option values one step back in time (at time $n - 1$) are computed by taking the present value of the expected future value of the option (13.20). Before stepping back another time increment, it is again necessary to see if any of the option values are below their early exercise value. Here is where dividends may enter the picture again. If no dividends are paid at time $n - 1$, then the early exercise value is simply the exercise price less the lattice stock price. If a dividend is paid at time $n - 1$, however, the early exercise proceeds equal the exercise price less the lattice stock price less the dividend. If any of the computed option values are below the exercise proceeds, they are replaced with the value of the exercise proceeds.

As we repeat the process and step back further in time, we must keep track of the sum of the present values of the dividends paid during the option's remaining life. At time $n - 1$, there was only one dividend and it was paid at time $n - 1$, so the sum equals the value of the single dividend paid at time $n - 1$. If we are at time $n - 2$ and there is a dividend paid at time $n - 2$ as well as a dividend paid at time $n - 1$, the sum of the present values of the promised dividends that should be included in the early exercise boundary check at time $n - 2$ is

$$PVD_{n-2} = D_{n-2} + \frac{D_{n-1}}{r^*}. \tag{13.22}$$

In other words, the early exercise boundary at time $n - 2$ is $X - S^x_{n-2,j} - PVD_{n-2}$. By the time the iterative procedure is complete, the early exercise boundary used to check the option price corresponding to the time 0 stock index level node will include the present value of all promised dividends as in equation (13.21).

EXAMPLE 13.2

Compute the value of an American-style put option that has an exercise price of \$50 and a time to expiration of 90 days. Assume the riskless rate of interest is 10 percent annually, the stock price is \$50, the standard deviation of the rate of return of the stock is 30 percent per year, and the stock pays a dividend of \$2 in exactly 60 days.

We proceed in two distinct steps. First we compute the European put option value using the formula (13.17), and then we compute the American put option value using the binomial method. In this way, we can identify the value of the early exercise premium on the put.

The current stock price net of the present value of the promised dividend is

$$S^x = 50 - 2e^{-.10(60/365)} = 48.033,$$

so the European put value can be computed as

$$p = 50e^{-.10(90/365)}N_1(-d_2) - 48.033N_1(-d_1),$$

where

$$d_1 = \frac{[\ln(48.033/50) + (.10 + .5(.30)^2)(90/365)]}{.30\sqrt{90/365}} = -.029$$

and $d_2 = -.029 - .149 = -.178$. The probabilities $N_1(.029)$ and $N_1(.178)$ are .512 and .571, so the European call value is

$$p = 48.782(.571) - 48.033(.512) = 3.26.$$

The value of the American put is computed using the binomial method. The number of time steps is set equal to 90, so the time increment Δt is one day or

Once the checks are performed, we go to column $n - 2$, repeat the steps and so on back through time. Eventually, we will work our way back to time 0, and the current value of the American put option (in column 0) will be identified.

To complete the binomial method illustration, suppose that the stock price lattice shown in Figure 13.4 underlies a 90-day American put option with an exercise price of 50. Applying the binomial method, the value of the American put is \$2.475. The value of this put using the European formula (13.16) is \$2.364, which means that the early exercise premium of the American put is worth about 11.1¢. Note that the value produced by the binomial method for the European put (by not checking for the early exercise constraints) is \$2.355, which is different from the \$2.364 obtained using the European formula. This is error due to the fact that the binomial method is only an approximation. In general, the accuracy of the binomial method increases with the number of time steps, holding other factors constant.

American Put Option on Dividend-Paying Stocks

The binomial method is also well suited to handle the case of valuing an American put option on a dividend-paying stock. If the dividends paid during the put's life are known with certainty, we first subtract the present value of the dividends from the current stock price, that is,

$$S_0^x = S_0 - \sum_{i=1}^{n} D_i e^{-rt_i}, \tag{13.21}$$

where D_i (t_i) is the amount of (time to) the i-th dividend paid during the option's life and S_0 is the current stock price. Next, we set up the binomial lattice, beginning with S_0^x rather than S_0. That is, if the current stock price net of dividends is S_0^x, the stock price at the end of the first time interval is either uS_0^x or dS_0^x. The values of u, d, and p are computed as (13.18a), (13.18b), and (13.18c).

With the stock price lattice (net of dividends) computed, the approximation method starts at the end of the option's life and works back to the present. At the end of the option's life, the option value at each stock price node is given by the intrinsic value of the put (13.19), where S^x replaces S. The option values one step back in time (at time $n - 1$) are computed by taking the present value of the expected future value of the option (13.20). Before stepping back another time increment, it is again necessary to see if any of the option values are below their early exercise value. Here is where dividends may enter the picture again. If no dividends are paid at time $n - 1$, then the early exercise value is simply the exercise price less the lattice stock price. If a dividend is paid at time $n - 1$, however, the early exercise proceeds equal the exercise price less the lattice stock price less the dividend. If any of the computed option values are below the exercise proceeds, they are replaced with the value of the exercise proceeds.

As we repeat the process and step back further in time, we must keep track of the sum of the present values of the dividends paid during the option's remaining life. At time $n - 1$, there was only one dividend and it was paid at time $n - 1$, so the sum equals the value of the single dividend paid at time $n - 1$. If we are at time $n - 2$ and there is a dividend paid at time $n - 2$ as well as a dividend paid at time $n - 1$, the sum of the present values of the promised dividends that should be included in the early exercise boundary check at time $n - 2$ is

$$PVD_{n-2} = D_{n-2} + \frac{D_{n-1}}{r^*}. \quad \textbf{(13.22)}$$

In other words, the early exercise boundary at time $n - 2$ is $X - S^x_{n-2,j} - PVD_{n-2}$. By the time the iterative procedure is complete, the early exercise boundary used to check the option price corresponding to the time 0 stock index level node will include the present value of all promised dividends as in equation (13.21).

EXAMPLE 13.2

Compute the value of an American-style put option that has an exercise price of \$50 and a time to expiration of 90 days. Assume the riskless rate of interest is 10 percent annually, the stock price is \$50, the standard deviation of the rate of return of the stock is 30 percent per year, and the stock pays a dividend of \$2 in exactly 60 days.

We proceed in two distinct steps. First we compute the European put option value using the formula (13.17), and then we compute the American put option value using the binomial method. In this way, we can identify the value of the early exercise premium on the put.

The current stock price net of the present value of the promised dividend is

$$S^x = 50 - 2e^{-.10(60/365)} = 48.033,$$

so the European put value can be computed as

$$p = 50e^{-.10(90/365)} N_1(-d_2) - 48.033N_1(-d_1),$$

where

$$d_1 = \frac{[\ln(48.033/50) + (.10 + .5(.30)^2)(90/365)]}{.30\sqrt{90/365}} = -.029$$

and $d_2 = -.029 - .149 = -.178$. The probabilities $N_1(.029)$ and $N_1(.178)$ are .512 and .571, so the European call value is

$$p = 48.782(.571) - 48.033(.512) = 3.26.$$

The value of the American put is computed using the binomial method. The number of time steps is set equal to 90, so the time increment Δt is one day or

.00274 years. The values of the factors u and d are $u = e^{.30\sqrt{.00274}} = 1.01583$, and $d = 1/.01583 = .98442$, with probabilities $p = (e^{.10(.00274)} - .98442)/(1.01583 - .98442) = .5048$ and $1 - p = .4952$, respectively. The possible values of the stock price (net of dividends) at the option's expiration range from 11.69 to 197.38. The value of the American put is 3.393, hence the early exercise premium on the American option is $3.393 - 3.262 = .131$.

Table 13.5 demonstrates how the value of the early exercise premium increases as the put option goes deeper in-the-money. At a stock price of \$40, for example, the early exercise premium is about 35¢, about 3 percent of the overall option value.

Finally, it is worthwhile to note that the dividend-adjusted binomial procedure outlined above not only handles an American put option on a dividend-paying stock but also handles American call options. Where the stock pays only a single dividend during the option's life, the American call option valuation equation (13.15b) is the most computationally efficient. Where the stock pays multiple dividends, however, the dividend-adjusted binomial method is much faster. We address this issue again when we value the American-style S&P 100 index options in the next chapter.

13.5 RIGHTS AND WARRANTS

Rights and warrants are securities issued by the firm. Usually, they are attached to debt offerings by the firm in order to reduce the coupon interest cost. Like call options, rights and warrants provide holders with the right to buy the underlying stock at a predetermined price within a specified period of time. Unlike call options, however, warrants are issued by the firm. Since the firm has a fixed amount of assets, the exercise of rights or warrants means that there will be more stockholders

TABLE 13.5 Simulated American and European put option values on a stock with a known discrete dividend. The put option has an exercise price of \$50 and a time to expiration of 90 days. The riskless rate of interest is 10 percent, and the standard deviation of the stock return is 30 percent. The stock pays a dividend of \$2 in 60 days.

Stock Price S	European Put Price $p(S,T;X)$	American Put Price $P(S,T;X)$	Early Exercise Premium ϵ_P
40	10.875	11.230	.355
45	6.510	6.757	.247
50	3.264	3.393	.129
55	1.360	1.406	.046
60	.476	.492	.016

sharing the same "pie," hence the value of existing shares will be diluted. Warrant valuation must account for this dilutionary effect.

In this section, we value rights and warrants explicitly recognizing the effects of dilution.[14] Effectively, there is little distinction between rights and warrants from a valuation standpoint. Rights tend to be short-term and at-the-money when they are issued; warrants tend to be long-term and out-of-the-money. For convenience, we use only the term "warrants" in the remaining part of this section.[15] The valuation approach is like that used in Chapter 11. Because a riskless hedge can be formed between the warrant and the value of the firm, we suffer no loss of generality and gain considerable mathematical tractability if we invoke an assumption of risk-neutrality.

Let S be the aggregate market value of the shares of the common stock currently outstanding; W, the aggregate market value of all warrants; r, the riskless rate of interest; and, V, the total market value of the firm. The firm is assumed to have only two sources of financing—stock and warrants—so the total market value of the firm may be defined as $V \equiv S + W$. The number of shares of stock outstanding is n_S, and n_W is the number of shares of stock sold if warrants are exercised. One warrant is assumed to provide the right to buy one share. The dilution factor possible due to the presence of the warrants is $\gamma \equiv n_W/(n_S + n_W)$. The stock is assumed to pay no dividends during the warrant's life. The standard deviation of the overall rate of return on the firm is σ. Finally, the warrant contract parameters are T, the time to expiration of warrants, and X, the aggregate amount paid by warrant holders to acquire the stock (i.e., the aggregate exercise price).

The assumptions used in the development of the warrant valuation equation are the same as those underlying the European call option except it is assumed the total market value of the firm is lognormally distributed at the warrants' expiration, *not* the firm's share price (i.e., $\ln(\tilde{V}_T/V)$ is normally distributed with variance σ^2).

Using risk-neutral valuation, the value of the firm's warrants today is the present value of the expected future terminal value, that is,

$$W = e^{-rT}E(\tilde{W}_T). \tag{13.23}$$

The terminal value of the warrants, in turn, is the proportion of the firm that the warrant holders will own if the warrants are exercised, γ, times the terminal value of the firm after the warrant holders pay the cash exercise amount, $\tilde{V}_T + X$, less the cash exercise amount, X, that is,

$$\tilde{W}_T = \begin{cases} \gamma(\tilde{V}_T + X) - X & \text{if} \quad \gamma(V_T + X) \geq X \\ 0 & \text{if} \quad \gamma(V_T + X) < X, \end{cases} \tag{13.24}$$

[14] The approach used here is based on Smith (1976).

[15] *Executive stock options*, a popular form of management incentive compensation, can be considered warrants from a valuation standpoint.

which can be rewritten

$$\tilde{W}_T = \begin{cases} \gamma\tilde{V}_T - (1-\gamma)X & \text{if} \quad \gamma V_T \geq (1-\gamma)X \\ 0 & \text{if} \quad \gamma V_T < (1-\gamma)X. \end{cases} \tag{13.25}$$

Note the similarity between the structure of the warrants payoffs in (13.25) and the European call option payoffs discussed in Chapter 11, that is,

$$\tilde{c}_T = \begin{cases} \tilde{S}_T - X & \text{if} \quad S_T \geq X \\ 0 & \text{if} \quad S_T < X. \end{cases} \tag{13.26}$$

In Chapter 11, we showed that the expected terminal value of the European call option for a non-dividend-paying common stock is

$$E(\tilde{c}_T) = e^{rT} SN(d_1) - XN(d_2), \tag{13.27}$$

where

$$d_1 = \frac{\ln(S/X) + (r + .5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T},$$

where S_T is lognormally distributed. In the warrant valuation problem, γV_T corresponds to the terminal stock price S_T and is assumed to be lognormally distributed. The term $(1 - \gamma)X$ is certain and corresponds to the exercise price of the stock option. It therefore follows that

$$E(\tilde{W}_T) = e^{rT}\gamma VN(d_1) - (1-\gamma)XN(d_2), \tag{13.28}$$

where

$$d_1 = \frac{\ln[\gamma V/((1-\gamma)X)] + (r + .5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Substituting (13.28) into (13.23), we find that the aggregate market value of the warrants of the firm is

$$W = \gamma VN(d_1) - e^{-rT}(1-\gamma)XN(d_2), \tag{13.29}$$

where

$$d_1 = \frac{\ln[\gamma V/((1-\gamma)X)] + (r + .5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

The *market value per warrant* is simply W from (13.29) divided by n_W.

One problem with applying (13.29) to value warrants is that it is difficult to estimate the volatility rate of the firm, σ. To estimate this parameter using historical data requires a time series of prices for both the stock and the warrant. Since the warrants may not actively trade, acquiring the historical price series for the warrant may be difficult. On the other hand, estimating the volatility rate of the stock, σ_S, is much easier since stocks are more actively traded and historical daily prices are available from a variety of sources. In addition, if the stock has listed options, the stock option pricing model can be used to compute the implied volatility rate of the stock.

Fortunately, the warrant valuation equation can be reformulated in terms of the stock's volatility rate rather than the volatility rate of the firm.[16] Since the rate of return on the stock is perfectly correlated with the rate of return on the overall firm,

$$\sigma_S = \eta_{SV}\sigma, \tag{13.30}$$

where η_{SV} is the elasticity of the stock price with respect to the value of the firm, that is, the percentage change in stock price for a given percentage change in the value of the firm. To estimate η_{SV}, first recall that by assumption the firm's value is the sum of the market value of the stock and the market value of warrants. Thus,

$$S = V - W.$$

The change in the market value of the stock for a given change in the value of the firm is, therefore,

$$\frac{\partial S}{\partial V} = 1 - \frac{\partial W}{\partial V}.$$

Second, from the valuation equation (13.29), we know that

$$\frac{\partial W}{\partial V} = \gamma N(d_1),$$

so

$$\frac{\partial S}{\partial V} = 1 - \gamma N(d_1). \tag{13.31}$$

[16] This idea was first suggested by Schulz and Trautmann (1991).

Finally, multiplying this term by V/S, we find that the elasticity of the stock price with respect to the value of the firm is

$$\eta_{SV} = \frac{\partial S/S}{\partial V/V} = [1 - \gamma N(d_1)]\frac{V}{S}. \tag{13.32}$$

Hence, to value the warrant as a function of the stock's volatility rate, we simply substitute the following term for the σ in (13.29):

$$\begin{aligned}\sigma &= \frac{1}{\eta_{SV}}\sigma_S \\ &= \frac{S}{[1 - \gamma N(d_1)]V}\sigma_S.\end{aligned} \tag{13.33}$$

Another somewhat perplexing consideration in applying (13.29) to value warrants is that the warrant value appears on both sides of the equation, directly on the left-hand side and indirectly through V (i.e., W is embedded in V) on the right-hand side. This poses no great difficulty. We simply find the value of W that satisfies the equation through some sort of numerical search procedure, just as we do when finding the yield to maturity of a coupon-bearing bond.

EXAMPLE 13.3

Compute the value of a one-year warrant whose exercise price is \$50. The current stock price is \$50, and the stock pays no dividends. The firm has only two sources of financing: 2,000 shares of stock and 500 warrants. One warrant entitles its holder to one share of common stock. Assume that the riskless rate of interest is 6 percent, and that the standard deviation of the rate of return on the firm is 30 percent.

The dilution factor posed by the warrants is

$$\gamma = \frac{500}{2,000 + 500} = .2,$$

the aggregate exercise proceeds to the firm are

$$X = 500 \times 50 = 25,000,$$

and the market value of the firm is

$$V = 2,000 \times 50 + W = 100,000 + W.$$

The *aggregate market value of the warrants* is computed by solving

$$W = .2(100,000 + W)N(d_1) - e^{-.06(1)}.8(25,000)N(d_2),$$

where

$$d_1 = \frac{\ln[.2(100,000 + W)/(.8(25,000))] + (.06 + .5(.30)^2)1}{.30\sqrt{1}}$$

and

$$d_2 = d_1 - .30\sqrt{1}.$$

The solution to this problem is obtained iteratively. The values of d_1 and d_2 on the final iteration are .4611 and .1611, respectively. The probabilities are $N(.4611) = .6776$ and $N(.1611) = .5640$. The aggregate market value of the warrants is $3,389.46, so the price per warrant is $6.78.

In the interest of completeness, it is worthwhile to note that the volatility rate of the stock in this exercise equals

$$\begin{aligned}\sigma_S &= \sigma\frac{[1 - \gamma N(d_1)]V}{S} \\ &= .30\frac{[1 - .2(.6776)]103,389.46}{100,000} \\ &= .2681.\end{aligned}$$

The volatility rate of the stock is lower than the volatility rate of the overall firm so the volatility rate of the warrants must be higher. Since the returns of the stock and the warrant are perfectly correlated,

$$\sigma = \sigma_S\left(\frac{S}{V}\right) + \sigma_W\left(\frac{W}{V}\right).$$

Therefore, the warrant volatility rate is

$$\sigma_W = \left[.3000 - .2661\left(\frac{100,000}{103,389.46}\right)\right]\left(\frac{103,389.46}{3,389.46}\right) = 1.3002.$$

13.6 SUMMARY

The focus of this chapter is stock options. Following a description of exchange-traded stock options in the first section, we adapt the general pricing principles of Chapters 10 and 11 to value call and put options on non-dividend-paying stocks. The principles are modified somewhat to account for the fact that common stocks typically pay discrete dividends during the option's life. We assume that the amount and the timing of the dividends paid during the option's life are known with certainty.

In this chapter, we also introduce the use of the binomial method to price American options. Although the specific application in this chapter is American-style options on stocks, the binomial method can be applied to the valuation of virtually any type of option. We use it again in Chapter 15, for example, to value interest rate options.

Finally, a special type of call option on common stock is considered. Specifically, firms often issue rights or warrants to raise new capital. Like call options, these contracts provide the holder with the right to buy the common shares of the firm at a fixed price within a specified period of time. Unlike call options, however, the firm sells (or gives away) the options, so, if the rights/warrants are exercised, the firm faces the prospect of having the equity of the firm diluted. The prospect of dilution has an important effect on warrant price.

APPENDIX 13.1

APPROXIMATION FOR THE CUMULATIVE BIVARIATE NORMAL DENSITY FUNCTION

In Appendix 11.2, a cumulative univariate normal density function approximation was provided to help compute the value of options on commodities with a constant, proportional cost-of-carry rate. In this chapter, we found that if a common stock pays a discrete dividend during the option's life, the American call option valuation equation requires the evaluation of a cumulative bivariate normal density function. While there are many available approximations for the cumulative bivariate normal distribution, the approximation provided here relies on Gaussian quadratures.[1] The approach is straightforward and efficient, and its maximum absolute error is .00000055.

The probability that x is less than a and that y is less than b for a standardized bivariate normal distribution is

$$\begin{aligned}\text{Prob}(x < a \text{ and } y < b) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{a} \int_{-\infty}^{b} exp\left[-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right] dx\ dy \\ &= N_2(a, b; \rho),\end{aligned}$$

where ρ is the correlation between the random variables x and y.

The first step in the approximation of the bivariate normal probability $N_2(a,b;\rho)$ involves developing a routine that evaluates the function $\phi(a,b;\rho)$ below:

$$\phi(a, b; \rho) \approx .31830989\sqrt{1-\rho^2} \sum_{i=1}^{5} \sum_{j=1}^{5} w_i w_j f(x_i, x_j), \tag{1}$$

where

$$f(x_i, x_j) = exp[a_1(2x_i - a_1) + b_1(2x_j - b_1) + 2\rho(x_i - a_1)(x_j - b_1)], \tag{2}$$

the pairs of weights (w) and corresponding abscissa values (x) are:

i, j	w	x
1	.24840615	.10024215
2	.39233107	.48281397
3	.21141819	1.0609498
4	.033246660	1.7797294
5	.00082485334	2.6697604

[1]The Gaussian quadrature method for approximating the bivariate normal is from Drezner (1978), and the Gaussian quadratures for the intergral are from Steen, Byrne, and Gelbard (1969). For a contingency table approach to this problem, see Wang (1987).

and the coefficients a_1 and b_1 are computed using

$$a_1 = \frac{a}{\sqrt{2(1-\rho^2)}} \quad \text{and} \quad b_1 = \frac{b}{\sqrt{2(1-\rho^2)}}.$$

The second step in the approximation involves computing the product, $ab\rho$.

If $ab\rho \leq 0$, compute the bivariate normal probability, $N_2(a,b;\rho)$, using the following rules:

1. If $a \leq 0$, $b \leq 0$, and $\rho \leq 0$, then $N_2(a,b;\rho) = \phi(a,b;\rho)$.
2. If $a \leq 0$, $b > 0$, and $\rho > 0$, then $N_2(a,b;\rho) = N_1(a) - \phi(a,-b;-\rho)$.
3. If $a > 0$, $b \leq 0$, and $\rho > 0$, then $N_2(a,b;\rho) = N_1(b) - \phi(-a,b;-\rho)$.
4. If $a > 0$, $b > 0$, and $\rho \leq 0$, then $N_2(a,b;\rho) = N_1(a) + N_1(b) - 1 + \phi(-a,-b;\rho)$.

If $ab\rho > 0$, compute the bivariate normal probability, $N_2(a,b;\rho)$, as:

$$N_2(a,b;\rho) = N_2(a,0;\rho_{ab}) + N_2(b,0;\rho_{ba}) - \delta,$$

where the values of $N_2(\cdot)$ on the right-hand side are computed from the rules for $ab\rho \leq 0$,

$$\rho_{ab} = \frac{(\rho a - b)\mathrm{Sgn}(a)}{\sqrt{a^2 - 2\rho ab + b^2}}, \quad \rho_{ba} = \frac{(\rho b - a)\mathrm{Sgn}(b)}{\sqrt{a^2 - 2\rho ab + b^2}},$$

$$\delta = \frac{1 - \mathrm{Sgn}(a) \times \mathrm{Sgn}(b)}{4},$$

and

$$\mathrm{Sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$$

$N_1(d)$ is the cumulative univariate normal probability. An approximation for $N_1(d)$ is contained in Appendix 11.2.

Finally, to assist those who may attempt to code this algorithm, sample computations for the bivariate normal probabilities are shown on page 340:

a	b	ρ	$N_2(a, b; \rho)$
-1.00	-1.00	-.50	.003782
-1.00	1.00	-.50	.096141
1.00	-1.00	-.50	.096141
1.00	1.00	-.50	.686472
-1.00	-1.00	.50	.062514
-1.00	1.00	.50	.154873
1.00	-1.00	.50	.154873
1.00	1.00	.50	.745203
.00	.00	.00	.250000
.00	.00	-.50	.166667
.00	.00	.50	.333333

EXAMPLE 13.A

Compute the risk-neutral probability that IBM and GM will have stock prices above \$120 and \$60, respectively, at the end of two months. The current price of IBM is \$107 and the current price of GM is \$48. Assume the riskless rate of interest is 10 percent annually, IBM and GM returns are bivariate normally distributed, IBM has a return volatility of 33 percent annually, GM has a return volatility of 36 percent annually, and the correlation between the returns of the two stocks is .6. Neither stock pays a dividend during the next two months.

The first step in finding this probability is to compute the upper integral limits for the standardized normal bivariate density function.

For IBM, the computation is

$$a = \frac{[\ln(107/120) + (.10 - .5(.33)^2)(2/12)]}{.33\sqrt{2/12}} = -.7948$$

and, for GM, the computation is

$$b = \frac{[\ln(48/60) + (.10 - .5(.36)^2)(2/12)]}{.36\sqrt{2/12}} = -1.4784.$$

The next step is to apply an approximation method to compute the bivariate normal probability. Applying the procedure described above, the probability is

$$N_2(a, b; \rho) = N_2(-.7948, -1.4784; .6) = .0463.$$

The probability that in two months IBM will have a stock price above 120 and that GM will have a stock price above 60 is slightly more than 4.6 percent.

It is instructive to note that the individual probabilities of each stock realizing its crtitical future value are

$$N_1(a) = N_1(-.7948) = .2134 \text{ and } N_1(b) = N_1(-1.4784) = .0697,$$

for IBM and GM, respectively. Thus, if the returns of IBM and GM were independent (i.e., their return correlation is 0), the probability that in two months IBM will have a stock price above 120 and that GM will have a stock price above 60 is .2134 $\times$ .0697 or about 1.49 percent. The reason that this probability is less than the 4.6 percent where the correlation is .6 is that, with high positive return correlation, an upward movement in IBM's stock price implies that GM's stock price will tend to move upward also. In the extreme case where these two stocks have perfect positive correlation (i.e., $\rho = +1$), the probability that in two months IBM will have a stock price above 120 and that GM will have a stock price above 60 is the lower of the two univariate probabilities, 6.97 percent.