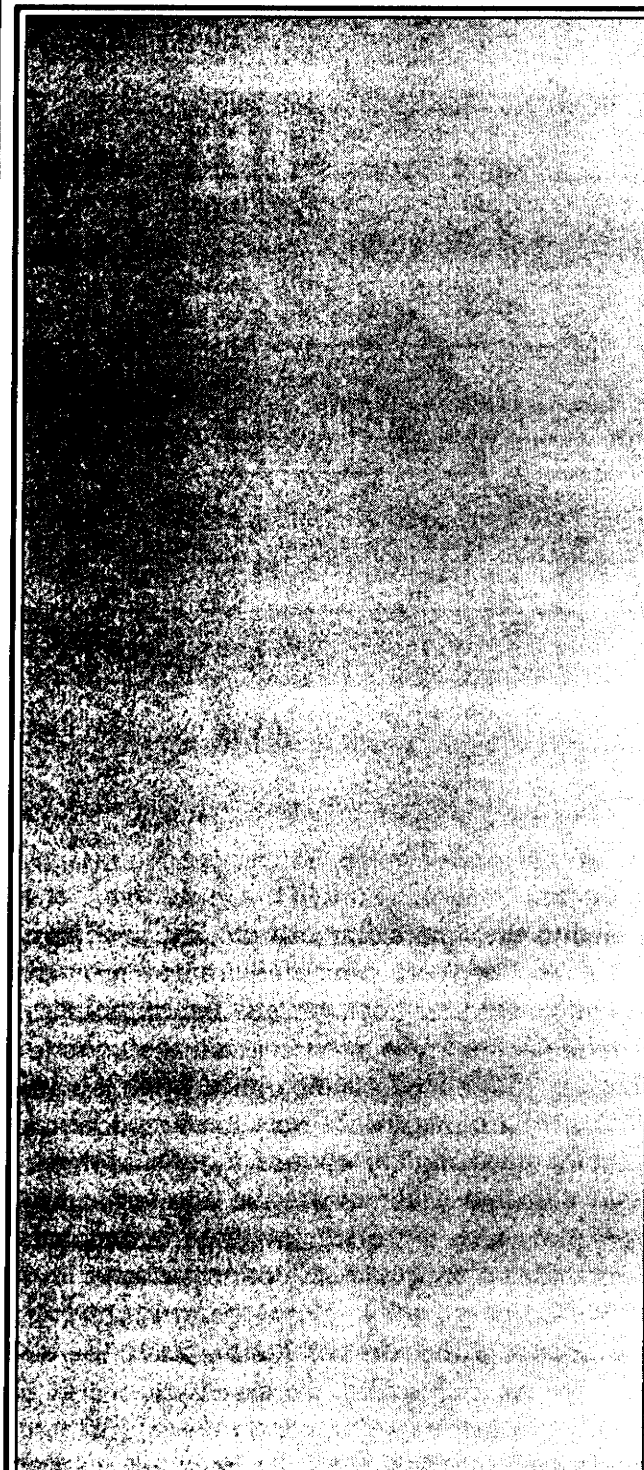
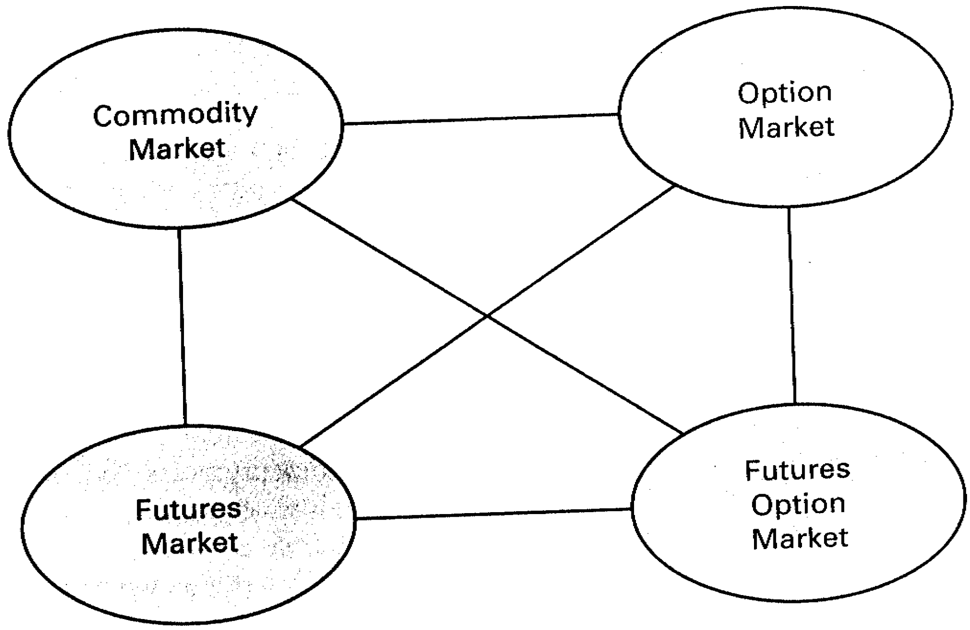
PART 3



OPTIONS

FIGURE 10.1 Interrelations Between Commodity Market and Markets for the Commodity's Derivative Instruments



prefixed by the nature of the commodity underlying the option contract, that is, a *stock option* is the right to buy or sell a common stock, a *foreign currency option* is the right to buy or sell a currency, a *bond option* is the right to buy or sell a bond, a *futures option* is the right to buy or sell a futures contract, and so on.

A contract that provides its holder with the right to buy the underlying commodity is called a *call option*; a contract that provides the right to sell is called a *put option*. In the option contract, the specified price at which the commodity may be bought or sold is called the *exercise price* or the *striking price* of the option. If the current commodity price exceeds the exercise price of the option, the call is *in-the-money* and the put is *out-of-the-money*. If the current commodity price is below the exercise price of the option, the call is *out-of-the-money* and the put is *in-the-money*. When the current commodity price is approximately equal to the exercise price of the options, both the call and the put are *at-the-money*.

Two different styles of option contracts trade in the U.S., *European* and *American* options. These option contracts are alike in all respects, except that European options may be exercised only at expiration, while American options may be exercised at any time up to and including the expiration day.

The notation that is most commonly used to represent parameters related to option pricing is as follows:

$S(\tilde{S}_T) \equiv$ current (random terminal) commodity price.

$F(\tilde{F}_T) \equiv$ current (random terminal) futures price.

$X \equiv$ exercise price or striking price.

$T \equiv$ time to expiration of the option.

$c(S, T; X) \equiv$ European call option with exercise price X and time to expiration T.

$p(S, T; X) \equiv$ European put option with exercise price X and time to expiration T.

$C(S, T; X) \equiv$ American call option with exercise price X and time to expiration T.

$P(S, T; X) \equiv$ American put option with exercise price X and time to expiration T.

Where the first argument in the parentheses of the option notation is S [e.g., $c(S, T; X)$], the option is a commodity option and the current commodity price is S. Where the first argument in the parentheses of the option notation is F [e.g., $c(F, T; X)$], the option is a futures option and the current futures price is F.

10.2. PROFIT DIAGRAMS AND VECTOR NOTATION

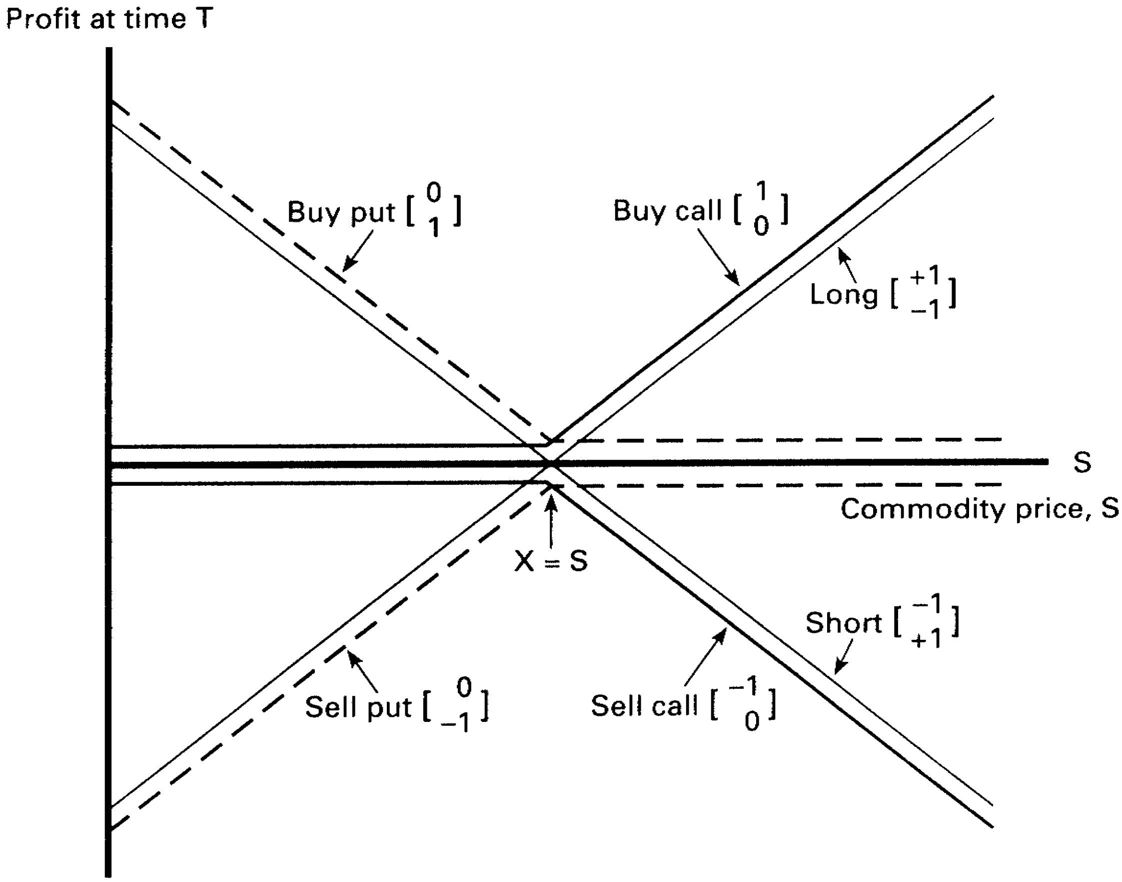
In Chapter 1, we illustrated the profits at expiration for different option positions. In this section, we introduce a simple vector notation that promotes understanding of the profit contingencies of complex option/commodity positions.

Figure 10.2 plots the profit at maturity, T, of different positions as a function of the price of the underlying commodity at maturity, S_T. (Option premiums are ignored in the current discussion.) We assume that all options have the same exercise price, X. Buying a call is profitable if $S_T > X$ and is not profitable if the commodity price fails to exceed the exercise price at maturity. The vector representation of this outcome is $\binom{1}{0}$. The first position in the vector indicates the dollar profit for every dollar that the commodity price at maturity exceeds the exercise price, and the second position indicates the dollar profit for every dollar that the commodity price at maturity falls below the exercise price. The vector notation for the seller of the call is $\binom{-1}{0}$. The seller of a call loses if $S_T > X$. The outcomes for the buyer and the seller of a put are also plotted in Figure 10.2. The vector notation for the buyer of a put is $\binom{0}{1}$ since the buyer makes money if the commodity price at maturity is below the exercise price. The vector notation for the seller of the put is $\binom{0}{-1}$ since the seller of the put loses if the commodity price at maturity falls below the exercise price.

Finally, the figure plots the profits to a long position and to a short position in the commodity, each initially established at the price, S, where $S = X$. We assume that no costs are incurred to store the commodity and that no dividends or other payments are made to the owners of the commodity. The vector notation for the long position is $\binom{1}{-1}$ and for the short position is $\binom{-1}{1}$.

The vector notation is useful for determining the profit to combinations of positions established at a single exercise price because corresponding positions in vectors may be added. For example, the equivalent of a long position in the commodity can be established by buying a call and selling a put: $\binom{1}{0} + \binom{0}{-1} = \binom{1}{-1}$. This is a *synthetic long*. Another way to see this is to add vertically the profits from the buy call and sell put positions in Figure 10.2. The equivalent of a short position, a *synthetic short*, can be established by selling a call and buying a put: $\binom{-1}{0} + \binom{0}{1} = \binom{-1}{1}$.

FIGURE 10.2 Profit at Maturity of Different Options Positions Ignoring the Initial Option Premiums



Just as options can be used to replicate positions in the underlying commodity, a position in the underlying commodity combined with one option can be used to replicate a position in the other option. A long position in a call, $\binom{1}{0}$, for example, can be replicated by buying the commodity and buying a put: $\binom{1}{-1} + \binom{0}{1} = \binom{1}{0}$. Similarly, a long position in a put, $\binom{0}{1}$, can be replicated by selling short the commodity and buying a call: $\binom{-1}{1} + \binom{1}{0} = \binom{0}{1}$. A frequent investment strategy is to sell a call against a long position in an underlying commodity, $\binom{-1}{0} + \binom{1}{-1}$, which yields the same profits as selling a put, $\binom{0}{-1}$.

A position that makes money whichever way the commodity price changes is a straddle. A straddle consists of buying a call and buying a put: $\binom{1}{0} + \binom{0}{1} = \binom{1}{1}$. Of course, such a position has a high price because the seller of the straddle must be compensated for the expected losses from selling the straddle. Someone who thinks the price of the underlying commodity is likely to rise might buy two calls. The vector notation for that is $\binom{2}{0}$ because the value of the options goes up two dollars for every dollar increase in the commodity price above the exercise price.

10.3 GENERAL DISCUSSION OF LOWER PRICE BOUNDS FOR OPTIONS

The lower price bound of an option is the lowest price at which the option could sell corresponding to each price of the underlying commodity. Before maturity, an option will usually sell for more than the lower bound because the option has a potential for profit that exceeds the potential for loss. In this section, we develop the intuition that underlies the structure of the lower price bounds for European and American call and put options. In the next section, we show the arbitrage portfolio transactions that ensure that each bound holds.

European Options

At maturity, the lower bounds of European options are given by the profit diagram of Figure 10.2. The call sells for its exercise value, that is, $c = \max[0, S_T - X]$ and the put sells for its exercise value, $p = \max[0, X - S_T]$. Before maturity, a European option sells for at least as much as the present value of its exercise value plus an allowance for the present value of the cost of storing the commodity. The price of the call option must satisfy the following condition for an in-the-money option:

$$c(S,T;X) \geq \frac{S(e^{bT}-1)}{e^{rT}} + \frac{S-X}{e^{rT}}, \tag{10.1}$$

where S is the price of the underlying asset at some time before maturity and T is the time until maturity.[1] If the commodity price at maturity were S, the exercise value of the call would be $S - X$. The value before maturity is the discounted value, the second term in (10.1). The call option must also reflect the storage costs associated with a long commodity position because in purchasing an option those storage costs are avoided. The present value of the storage costs, including the interest cost of the funds tied up at S, is the first term of (10.1).

Since the option could be out-of-the-money, the full lower bound condition, which is plotted in Figure 10.3, is

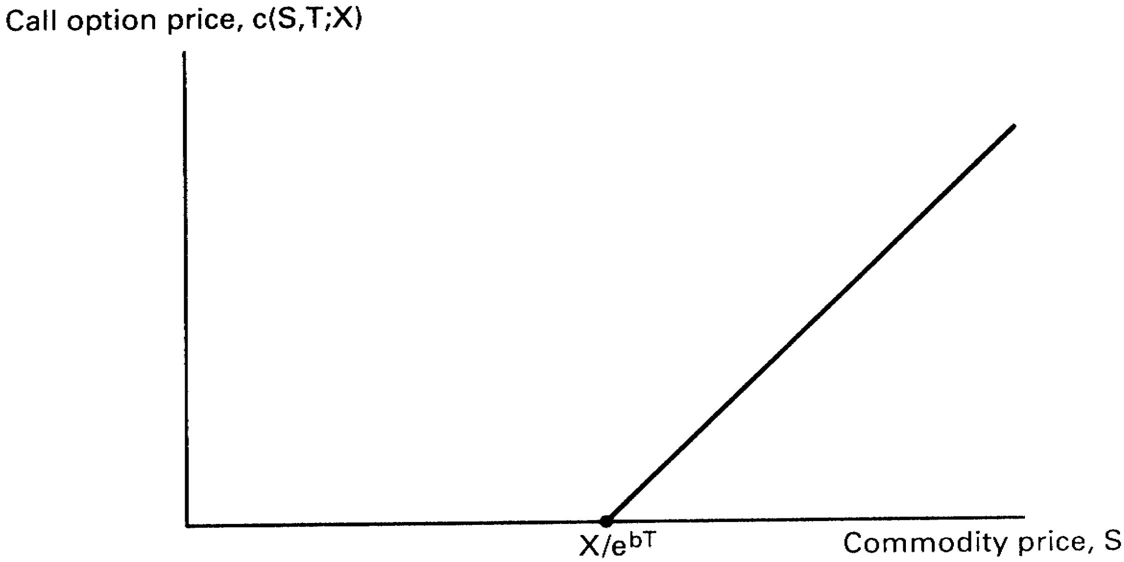
$$c(S,T;X) \geq \max[0, \frac{S(e^{bT}-1)}{e^{rT}} + \frac{S-X}{e^{rT}}]. \tag{10.2a}$$

The condition states that a European call option can never sell for less than the amount given in the right-hand side of (10.2a). Correspondingly, the lower bound for a European put is

$$p(S,T;X) \geq \max[0, \frac{-S(e^{bT}-1)}{e^{rT}} + \frac{X-S}{e^{rT}}]. \tag{10.2b}$$

[1] The right-hand side of (10.1) can be simplified to $Se^{(b-r)T} - Xe^{-rT}$, which will be done later in the chapter.

FIGURE 10.3 Lower Price Bound of European Call Option



$$\text{For } S > X/e^{bT},\ c(S,T;X) = \frac{S(e^{bT}-1)}{e^{rT}} + \frac{S-X}{e^{rT}}.$$

For non-zero put values, the lower bound is the present value of the exercise value of the put plus an adjustment for storage costs.

American Options

The distinguishing feature of American options, in contrast to European options, is that they can be exercised early. Because the right to early exercise cannot have a negative value (i.e., you do not have to be paid to be induced to take on a privilege), the following two conditions apply:

$$C(S,T;X) \geq c(S,T;X) \tag{10.3a}$$

and

$$P(S,T;X) \geq p(S,T;X). \tag{10.3b}$$

These conditions do not say that the American options *will* have greater values than their European counterparts—only that they *will not* have lower values. The right to early exercise may not have positive value, but it will never have negative value.

Since American options cannot sell for less than European options, a lower bound of American options is the lower bound of the corresponding European option with the same exercise price and maturity. An American option has the additional benefit that it may be exercised immediately to receive the exercise value—$S - X$ for the American call and $X - S$ for the American put. This means the lower bound for the American option is the lower bound of the European option or the current exercise value, whichever is higher. The lower bounds are

$$C(S,T;X) \geq \max[0, \frac{S(e^{bT}-1)}{e^{rT}} + \frac{S-X}{e^{rT}}, S-X], \tag{10.4a}$$

and

$$P(S,T;X) \geq \max[0, \frac{-S(e^{bT}-1)}{e^{rT}} + \frac{X-S}{e^{rT}}, X-S]. \tag{10.4b}$$

10.4 ARBITRAGE PROOFS OF LOWER PRICE BOUNDS

Before proceeding with developments of rational price bounds on commodity options, it is worthwhile to reintroduce the concept of a rollover position. Recall that in Chapter 3 we introduced a storage fund that was invested in the commodity. The sum of the storage fund and the position in one unit of the commodity is called a rollover position in the commodity. In Chapter 3, we used a portfolio consisting of a short futures position and a long rollover position in the underlying commodity to demonstrate basis arbitrage. In this chapter, we use the rollover commodity position to demonstrate the links between the price of an option and the commodity underlying the option. A long rollover position in the underlying commodity begins with an investment of $e^{(b-r)T}$ in the underlying commodity at the end of day 0. At the end of each subsequent day, the position is reduced (increased) by the factor $e^{-(b-r)}$ if $b > r$ ($b < r$). At the end of day T, exactly one unit of the commodity is on hand. It is also worthwhile to point out that a futures contract is a commodity that costs nothing to carry, that is, the cost-of-carry rate, b, equals zero. A rollover futures position, therefore, begins with e^{-rT} futures contracts at the end of day 0 and increases by the factor e^{r} each day.

European Call Option

The lower price bound of a European call option can be determined by considering the initial and terminal values of a portfolio that consists of a long position in the European call option $c(S, T; X)$, a long position of Xe^{-rT} riskless bonds, and a short rollover position in the commodity at price $Se^{(b-r)T}$, as illustrated in Table 10.1.[2] The amount of riskless lending is determined by the fact that we need X on hand at time T in order to exercise the call. If, at the option's expiration, the commodity price exceeds the exercise price, the call is exercised and the one unit of the underlying commodity received is used to cover the short commodity position. To pay for exercising the call, the riskless bonds are used exactly. The net terminal value of the portfolio is therefore 0. On the other hand, if the commodity price is below the exercise price at expiration, the call expires worthless and the value of the riskless bonds exceeds the value necessary to cover the short commodity position. In

[2] Note that the arguments of the option price notation in the table have been suppressed for convenience.

TABLE 10.1 Arbitrage transactions for establishing lower price bound of European call option.

$$c(S,T;X) \geq Se^{(b-r)T} - Xe^{-rT}$$

Position	Initial Value	Terminal Value $\tilde{S}_T \leq X$	Terminal Value $\tilde{S}_T > X$
Buy European call	$-c$	0	$\tilde{S}_T - X$
Buy riskless bonds	$-Xe^{-rT}$	X	X
Sell rollover position in commodity	$Se^{(b-r)T}$	$-\tilde{S}_T$	$-\tilde{S}_T$
Net portfolio value	$-c + Se^{(b-r)T} - Xe^{-rT}$	$X - \tilde{S}_T$	0

this case, the net terminal portfolio value is positive. Since the portfolio holder is assured to have a nonnegative terminal value to his portfolio, the initial value must be nonpositive, that is,

$$-c(S,T;X) + Se^{(b-r)T} - Xe^{-rT} \leq 0$$

or

$$c(S,T;X) \geq Se^{(b-r)T} - Xe^{-rT}, \tag{10.5a}$$

otherwise, costless arbitrage profits are possible. This condition is the same as (10.1), which can be seen by adding and subtracting Se^{-rT} on the right hand side of (10.5a).

Condition (10.5a) shows only one of the lower price bounds of the European call. In a rational market, the option price will never be negative since it is a right rather than an obligation. The complete lower price bound condition for the European call is, therefore,

$$c(S,T;X) \geq \max[0, Se^{(b-r)T} - Xe^{-rT}], \tag{10.6a}$$

which can be shown to be the same as (10.2a).

American Call Option

Since the American option always sells for more than the corresponding European option, the American call option is bounded from below by (10.6a). It is also bounded from below by the proceeds from exercising the call immediately, $S - X$. Otherwise, costless arbitrage profits could be earned by buying the call and exercising it immediately. The complete lower price bound condition for the American call option is, therefore,

$$C(S,T;X) \geq \max[0, Se^{(b-r)T} - Xe^{-rT}, S - X], \tag{10.7a}$$

which can be shown to be the same as (10.4a).

If $b \geqslant r$, the second term in the brackets of (10.7a) exceeds the third, which means that the lower bound of the American call is $Se^{(b-r)T} - Xe^{-rT}$, the European lower bound. In other words, when $b \geqslant r$, an American call behaves like a European call; it will not be exercised before maturity. One can write the condition that the second term in the brackets of (10.7a) exceeds the third term in the brackets as follows:

$$S < X\left[\frac{1-e^{-rT}}{1-e^{(b-r)T}}\right]. \tag{10.8}$$

If this condition is met, the American call will have the same lower bound as the European call. The condition, (10.8) will be met if $b \geqslant r$ (keeping in mind that we are looking at values of S for which the lower bound of the call option exceeds zero). On the other hand, for values of $b < r$, the second term in the brackets of (10.7a) can be less than the third term with the result that (10.8) is not satisfied. This implies that the lower bound of the American call is $S - X$. Options that do not satisfy (10.8) may be exercised early and will have a value greater than the corresponding European option.

European Put Option

The lower price bound for a European put option can be derived by considering the initial and terminal values of a portfolio that consists of a long position in the European put option, $p(S, T; X)$, a long rollover position in the underlying commodity, $Se^{(b-r)T}$, and a short position of Xe^{-rT} of riskless bonds, as is done in Table 10.2. The amount of the riskless borrowing is determined by the fact that exercising the put at expiration will provide X. Here, if the commodity price is below the exercise price at the put option's expiration, the commodity on hand is sold for the exercise price of the put by exercising the put option. The exercise proceeds are then used to repay the riskless borrowing. The net effect is that the terminal value of the portfolio will be equal to 0. On the other hand, if the commodity price exceeds the exercise price at expiration, the net portfolio value will be positive because the put option expires worthless and the commodity price exceeds the amount necessary to repay the riskless borrowing. Since this portfolio provides a nonnegative terminal value, it must be the case that the initial value is nonpositive. If the net initial value of the portfolio is nonpositive, that is, if

$$-p(S,T;X) - Se^{(b-r)T} + Xe^{-rT} \leq 0,$$

TABLE 10.2 Arbitrage transactions for establishing lower price bound of European put option.

$$p(S,T;X) \geq Xe^{-rT} - Se^{(b-r)T}$$

Position	Initial Value	Terminal Value $\tilde{S}_T \leq X$	Terminal Value $\tilde{S}_T > X$
Buy European put	$-p$	$X - \tilde{S}_T$	0
Buy rollover position in commodity	$-Se^{(b-r)T}$	$\tilde{S}_T$	$\tilde{S}_T$
Borrow Xe^{-rT}	Xe^{-rT}	$-X$	$-X$
Net portfolio value	$-p - Se^{(b-r)T} + Xe^{-rT}$	0	$\tilde{S}_T - X$

then

$$p(S,T;X) \geq Xe^{-rT} - Se^{(b-r)T}. \tag{10.5b}$$

Adding the nonnegativity constraint of the European put option value,

$$p(S,T;X) \geq \max[0, Xe^{-rT} - Se^{(b-r)T}], \tag{10.6b}$$

which can be shown to be the same as (10.2b).

American Put Option

Naturally, the lower price bound condition for the European put also applies to the American put. But, the exercisable proceeds of the American put, $X - S$, may be greater than $Xe^{-rT} - Se^{(b-r)T}$, and we know that in a rationally functioning market the American put is bounded from below by $X - S$. Otherwise, costless arbitrage profits could be earned by buying the put option and immediately exercising it. So, the complete lower bound condition for the American put is

$$P(S,T;X) \geq \max[0, Xe^{-rT} - Se^{(b-r)T}, X - S], \tag{10.7b}$$

which is the same as (10.4b).

It is worthwhile to note that at least part of the lower price bound of the European put lies to the left of the exercisable proceeds of the American put, independent of whether or not the cost-of-carry rate is greater than or less than the riskless rate of interest. (This is shown in Figure 10.5 on page 189.) The implication is that, for commodity options, it cannot be said that the American put option will never be exercised early.

TABLE 10.3 Summary of lower price bounds for European and American commodity and futures options.

Option Type	Commodity Option	Futures Option
European call	$\max[0, Se^{(b-r)T} - Xe^{-rT}]$	$\max[0, (F - X)e^{-rT}]$
American call	$\max[0, Se^{(b-r)T} - Xe^{-rT}, S - X]$	$\max[0, F - X]$
European put	$\max[0, Xe^{-rT} - Se^{(b-r)T}]$	$\max[0, (X - F)e^{-rT}]$
American put	$\max[0, Xe^{-rT} - Se^{(b-r)T}, X - S]$	$\max[0, X - F]$

Futures Options

Table 10.3 contains a summary of the lower price bound conditions for commodity and futures options. Recall that earlier in this chapter we noted that a futures contract is a commodity with a zero cost-of-carry rate. The price bounds in the last column of the table make this substitution.

10.5 EARLY EXERCISE

Earlier, we noted that the right to early exercise of American options is a privilege and must have a nonnegative value. If we consider the value of American options to be the sum of their European counterparts plus their respective early exercise premiums, ϵ_C and ϵ_P, that is,

$$C(S,T;X) = c(S,T;X) + \epsilon_C(S,T;X) \tag{10.9a}$$

and

$$P(S,T;X) = p(S,T;X) + \epsilon_P(S,T;X), \tag{10.9b}$$

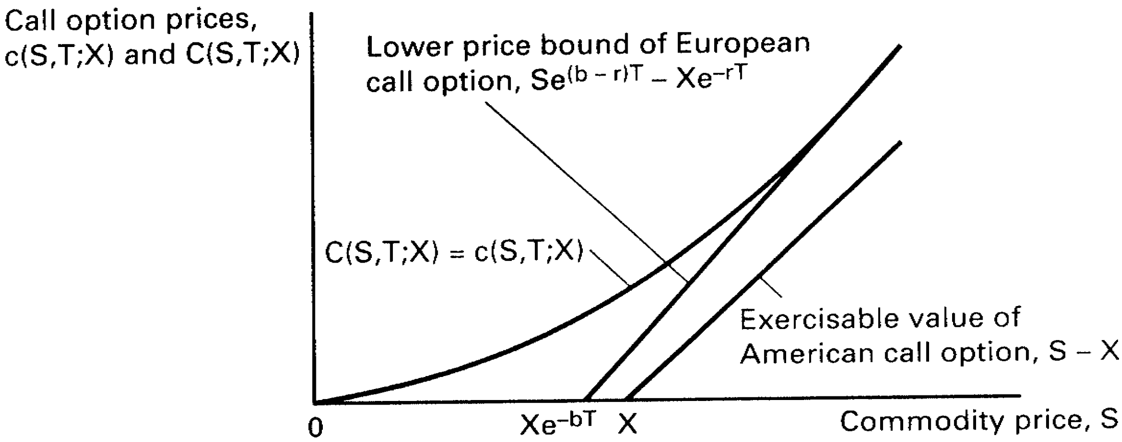
it is obvious that the American options sell for at least as much as their European counterparts.

That is not to say, however, that the American options will always be exercised early, or even that they may be. In the case of options on non-dividend-paying stocks, for example, the American call will never be exercised early.

American Call Option

Whether an American call option written on a commodity may be exercised early or not is contingent on the cost-of-carry rate, b. If $b \geqslant r$, the American call will not be exercised early. To see this, consider the lower price bound condition (10.7a) or, alternatively, Figure 10.4a. If $b \geqslant r$, the exercisable proceeds of the call, S −

FIGURE 10.4a European and American call option prices as a function of the underlying commodity price when the cost-of-carry rate (b) exceeds the riskless rate of interest (r) so the American call will not be optimally exercised early.



X, are always less than the minimum value for which the call will trade in the marketplace, $Se^{(b-r)T} - Xe^{-rT}$. Since the American call is worth more unexercised or "alive" than exercised or "dead," it will never be exercised prior to expiration. Thus, if $b \geqslant r$, the early exercise premium of the American call is worth nothing, that is,

$$\epsilon_C(S,T;X) = 0; \tag{10.10}$$

and, from equation (10.9a), the American call option has a value equal to the European call option,

$$C(S,T;X) = c(S,T;X). \tag{10.11}$$

The intuition underlying the fact that the American call will not be exercised early when $b \geqslant r$ can be developed most easily by considering the minimum amount lost by early exercise, that is,

$$S - X - [Se^{(b-r)T} - Xe^{-rT}]. \tag{10.12}$$

Rearranging (10.12) to isolate terms on S and X, we get

$$S[1 - e^{(b-r)T}] - X(1 - e^{-rT}). \tag{10.13}$$

Expression (10.13) says the following: If the American call option is exercised now instead of at expiration, the American call holder loses in two ways. First, he incurs the present value of the storage costs that he will pay as a result of taking delivery of the underlying commodity, $S[1 - e^{(b-r)T}]$.[3] By holding the option, the option holder does not have direct investment in the underlying commodity—only the right to buy the commodity in the future. If he takes delivery now he faces the prospect of storing the commodity, insuring it, etc. Second, he incurs the present value of the interest foregone on the exercise price of the option, $X(1 - e^{-rT})$. If the option is exercised now, the option holder is obliged to make payment in the amount X now instead of later, thereby foregoing the interest income that he could earn on the exercise price of the option. Figure 10.4a conveniently summarizes these effects by showing that the lower price bound of the European call option exceeds the exercisable proceeds of the American call option for all plausible values of the option price.

In the case where the cost-of-carry rate, b, is less than the riskless rate of interest, r, it may be optimal to exercise the American call early, as is seen by examining expression (10.13). When $b < r$, there are offsetting influences affecting the decision about early exercise. On one hand, deferring early exercise allows the call option holder to implicitly earn interest on the exercise price of the option, as was noted above. On the other hand, $b < r$ means that some form of yield accrues to the holder of the underlying commodity. For example, suppose that the call is written on a stock index and that the stock index portfolio accrues dividends at a known rate.[4] Deferring exercise means interest income is being earned, but dividend yield is being foregone. Note that the larger the value of the current commodity price S, the larger the value of expression (10.13), and the higher the profitability of early exercise. This fact is reflected in Figure 10.4b, through the increasing distance between the exercisable value of the American call and the lower price bound of the European call as the underlying commodity price grows larger. At any level of commodity price, however, there is a non-zero benefit to early exercise if $b < r$, so the early exercise premium has positive value,

$$\epsilon_C(S,T;X) > 0, \tag{10.14}$$

and the American call option has a value greater than the European call option,

$$C(S,T;X) > c(S,T;X). \tag{10.15}$$

[3] Because $b \geqslant r$, this term is nonpositive.

[4] Recall that the cost-of-carry rate, b, consists of the interest rate, r, plus the cost of storage, insurance, etc. Because the only "cost" other than interest involved in carrying a stock portfolio is the dividend yield (i.e., a negative cost), $b < r$.

FIGURE 10.4b European and American call option prices as a function of the underlying commodity price when the cost-of-carry rate (b) is less than the riskless rate of interest (r) so the American call may be optimally exercised early.

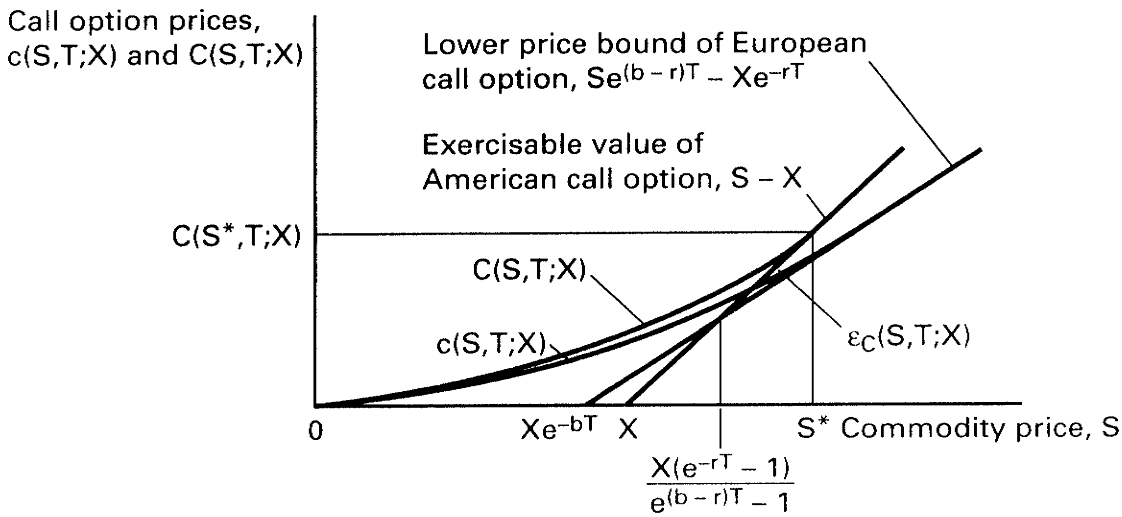


Table 10.4 contains an example of a call option for which early exercise can be optimal. The underlying commodity in this example is a foreign currency. The U.S interest rate is assumed to be 8 percent annually and the foreign interest rate is assumed to be 12 percent annually. As a result, the cost of carry is negative, that is, −4 percent annually. The exercise price of the call option is 150 cents and the maturity is 30 days. In this example, early exercise occurs if the price of the currency reaches 165 cents. At this point, the American option price equals the American lower bound because the option is being priced on the assumption that it will be exercised. Early exercise is desirable because taking possession of the currency and investing the currency in the foreign country provides a relatively higher rate of interest income (12 percent versus 8 percent domestically). If the option were held to maturity, its value is only 14.5692 (i.e., the European option value). It is worth noting that, for currency prices from 161 to 164, the American lower bound *exceeds* the price of the European option but early exercise is not optimal. In this range, the market price of the American option exceeds the American option lower bound, hence early exercise is not optimal.

American Put Option

Condition (10.7b), as well as Figure 10.5, show that there always exists an opportunity that any American put can be optimally exercised early. There is always some region of commodity prices over which the exercisable proceeds of the put will be higher than the lower price bound condition of the European put option. To gain intuition about the nature of the tradeoff involved here, difference the exercisable proceeds of the American put and the lower price bound of the European put, as we did for the American call in expression (10.13). The difference is

TABLE 10.4 Lower bounds and prices for European and American foreign currency call options. The option exercise price (X) is 150, and time to expiration (T) is 30 days (.08219 years). The domestic riskless rate of interest (r_d) is 8 percent, and the foreign riskless rate of interest (r_f) is 12 percent. The cost-of-carry rate (b) is therefore −4 percent. (The underlying currency has an annual volatility rate of 20 percent.)

Currency Price	European Lower Bound	American Lower Bound	European Option Price	American Option Price
150	0.0000	0.0000	3.1637	3.1991
151	0.5011	1.0000	3.6704	3.7125
152	1.4913	2.0000	4.2224	4.2726
153	2.4814	3.0000	4.8189	4.8784
154	3.4716	4.0000	5.4581	5.5288
155	4.4618	5.0000	6.1381	6.2219
156	5.4520	6.0000	6.8564	6.9556
157	6.4422	7.0000	7.6103	7.7276
158	7.4324	8.0000	8.3969	8.5355
159	8.4226	9.0000	9.2132	9.3767
160	9.4127	10.0000	10.0560	10.2489
161	10.4029	11.0000	10.9225	11.1496
162	11.3931	12.0000	11.8097	12.0769
163	12.3833	13.0000	12.7149	13.0289
164	13.3735	14.0000	13.6355	14.0041
165	14.3637	15.0000	14.5692	15.0000
166	15.3538	16.0000	15.5138	16.0000
167	16.3440	17.0000	16.4677	17.0000
168	17.3342	18.0000	17.4291	18.0000
169	18.3244	19.0000	18.3967	19.0000
170	19.3146	20.0000	19.3693	20.0000

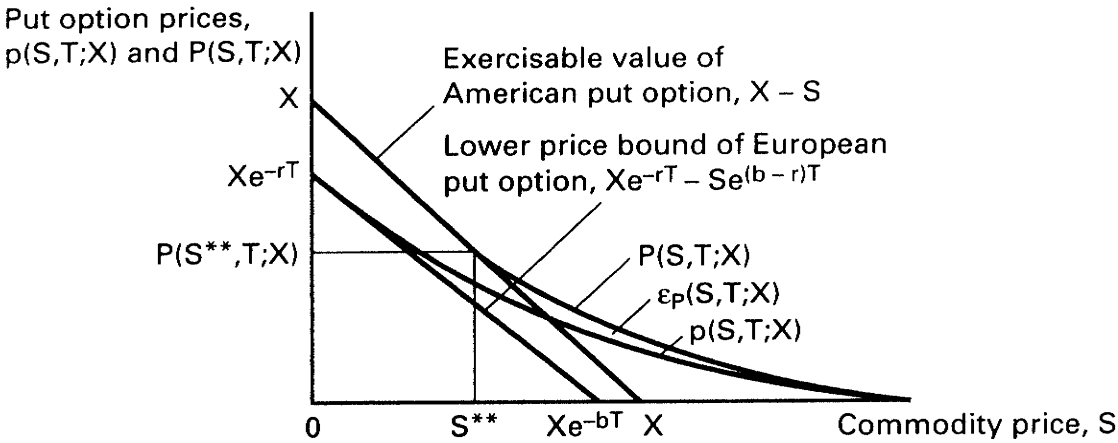
$$X - S - [Xe^{-rT} - Se^{(b-r)T}] \tag{10.16}$$

or, more simply,

$$X(1 - e^{-rT}) - S[1 - e^{(b-r)T}]. \tag{10.17}$$

The first term of (10.17) is the present value of the interest that can be earned if the option is exercised immediately. If the option holder exercises his put, he receives X and delivers the underlying commodity worth S.

FIGURE 10.5 European and American Put Option Prices as a Function of the Underlying Commodity Price



The proceeds from exercise can be invested immediately to earn interest. The net effect of the second term may be positive or negative, depending on whether $b > r$ or $b < r$. In the former case, exercising the put means the option holder can deliver the underlying commodity and forego the storage costs involved with deferring exercise. In the latter case, exercising the put means the option holder will be delivering a commodity that is currently providing her with some form of yield. She may be reluctant to do so, but expression (10.17) will be unambiguously positive for cases where the yield on the commodity is less than the riskless rate of interest.

The prospect of early exercise of the American put may also be seen geometrically in Figure 10.5. Independent of the value of b, there is always some range of commodity prices over which the exercisable proceeds of the American put, $X - S$, are greater than those of the European put option; therefore, there is always some possibility that the American put will be exercised early. To see that this is the case, consider what happens if the commodity price falls to zero. The value of the American put equals the exercise price of the put since the American put option holder has the right to sell a commodity with price zero for X at any time. In fact, in the event that the commodity price falls to zero, the American put option holder exercises his option immediately because (a) he can start earning interest on the proceeds from exercise, and (b) the commodity price may rise in which case the put price will fall. At $S = 0$, however, the European put option has a value of Xe^{-rT}. To recognize this, consider the boundary conditions on the put's price. The lower price bound is given by condition (10.2b). At $S = 0$, the minimum value for the put option is Xe^{-rT}. On the other hand, because the commodity price cannot be less than zero at the option's expiration, the present value of the maximum exercise proceeds is Xe^{-rT}. If the European put option's price is bounded from above and below by Xe^{-rT}, it follows that its price is Xe^{-rT}. Because the American put can be exercised immediately for proceeds equal to X while the European put has a lesser value, the early exercise premium must be positive for $S = 0$. In general, as long as there is some chance of early exercise, the early exercise premium has positive value,

$$\epsilon_P(S, T; X) > 0 \tag{10.18}$$

and the American put is worth more than the European put,

$$P(S, T; X) > p(S, T; X). \tag{10.19}$$

American Futures Options

Unlike the American option on the underlying commodity, there are no conditions under which it can be said that the American call or put option on a futures contract will not be exercised early. One can think of a futures contract as a commodity with a zero carrying cost ($b = 0$) and analyze the minimum loss from early exercise for calls, (10.13), and for puts, (10.17). When $b = 0$, both these expressions are positive, which means that early exercise of an in-the-money American futures option might be profitable.

The intuition for the possible early exercise of American futures options is straightforward. Consider a call option on a futures contract. If the call is exercised, a long futures position is established for the call buyer at the exercise price of the call. But payment of the exercise price is not required just as payment of the futures price is not required when a futures contract is entered into; only profits and losses are paid. If the option is in the money, profits are paid to the call buyer upon exercise. By way of example, consider a call on wheat futures at $X = \$3.00$ per bushel, and assume the current futures price is \$3.50 per bushel. If a call option on one futures contract is exercised, the call buyer assumes a futures position at a futures price of \$3.00. Since the current price is \$3.50, he is paid the profit of 50 cents per bushel, or \$2,500 on a contract of 5,000 bushels. These profits can be invested immediately to earn interest. At the same time, ownership of the futures contract does not impose the carrying cost that would be incurred if the underlying commodity were owned. Early exercise might be desirable because it allows profits to be invested sooner. Of course, the desirability of early exercise is offset, as usual, by the loss of the downside protection provided by the option. The intuition is the same for a futures put option. By exercising a put option on a commodity contract that has fallen in price, profits can be received early and interest earned.

10.6 GENERAL DISCUSSION OF PUT-CALL PARITY

Put-call parity refers to the relation between the price of a put and the price of a call with the same exercise price, expiration date, and underlying commodity price. We showed earlier that the outcome at maturity from the purchase of a call option could be replicated by a long position in the commodity plus the purchase of a put. Since one can buy a call directly or indirectly by purchasing a put and taking a long position, the prices of puts and calls must clearly be related. We restrict our discussion here to European options that are held to maturity.

Option *converters* take advantage of discrepancies in the prices of puts and calls. If call prices are too high, they sell calls, buy puts, and go long the underlying commodity. The vector notation for the resulting position is

$$\binom{-1}{0} + \binom{0}{1} + \binom{1}{-1} = \binom{0}{0}.$$

Selling a call and buying a put is a synthetic short position. By going long the underlying commodity, a perfect hedge is established, with the result that however the commodity price changes, no losses or gains result. Since the ending position is riskless, the profits to converters must be zero in equilibrium. We assume first that the exercise price of the options, X, is equal to the current price of the commodity, S. In that case, the present value of the cash flows associated with the hedged position is the revenue from selling the call, $c(S, T; X)$, the cost of the put, $-p(S, T; X)$, and the present value of the cost of going long the commodity, $-S(e^{bT} - 1)/e^{rT}$. The cost of going long includes the interest cost of funds tied up plus any storage costs less any income payments. Equilibrium requires the sum of these cash flows to be zero:

$$c(S,T;X) - p(S,T;X) - \frac{S(e^{bT}-1)}{e^{rT}} = 0,$$

or

$$c(S,T;X) - p(S,T;X) = \frac{S(e^{bT}-1)}{e^{rT}}. \tag{10.20}$$

If put prices were high relative to call prices, converters would sell puts, $\binom{0}{-1}$, buy calls, $\binom{1}{0}$, and take a short position in the underlying commodity, $\binom{-1}{1}$, to establish a perfect hedge. Under the assumption that the short can earn the storage fees that the long pays, the same equilibrium relation results.

A slight adjustment to the put-call parity relation is required if the price at which the position in the underlying commodity is established differs from the exercise price of the options. If $S \neq X$, the put-call parity relation for European options is

$$c(S,T;X) - p(S,T;X) = \frac{S(e^{bT}-1)}{e^{rT}} + \frac{S-X}{e^{rT}}, \tag{10.21}$$

where the last term accounts for the call or the put being in the money. While the put-call parity equation, (10.21), looks formidable, the basic idea is very simple. The put-call parity relation simply says that the call price minus the put price is the present value of the cost of holding the underlying commodity until maturity of the options plus the present value of the amount by which the commodity price exceeds the exercise price. We assume that storage costs are incurred at a continuous rate, b, but other assumptions could be made. For example, holding costs might be paid at the beginning as a lump sum, B. In that case, the first term on the right-hand side of (10.21) would be B.

To illustrate put-call parity for European options, consider the example in Table 10.4. Specifically, when the underlying currency price is 155, the European call price is 6.1381. On the basis of this price, the European put price can be computed as

$$6.1381 - p(S, T; X) = \frac{155(e^{(-.04)(.08219)} - 1) + 155 - 150}{e^{(.08)(.08219)}} = 4.4618.$$

Therefore, the value of the European put implied by put-call parity is 1.6763.

The put-call parity relation established above may not hold exactly for American options because early exercise of an American option can break up the riskless hedge. For example, the converter who sells puts, buys calls, and goes short the underlying commodity, may have the put exercised. The commodity delivered to the converter at the exercise price can be used to pay back the commodity borrowed for the short sale, but the converter must also liquidate the investment of the proceeds of the short sale (which are needed to pay for the commodity delivered) and that might be done at a loss. As a result, for American options, one can only establish bounds on the difference between call and put prices, something that is done in the next section.

10.7 ARBITRAGE PROOFS OF PUT-CALL PARITY RELATIONS

European Options

The put-call parity relation, (10.21), established above for European commodity options, can also be written as

$$c(S, T; X) - p(S, T; X) = Se^{(b-r)T} - Xe^{-rT}. \qquad \textbf{(10.22)}$$

To understand how this relation is derived, consider an investment portfolio that consists of selling a European call, buying a European put at the same exercise price, buying a rollover position in the underlying commodity beginning with $e^{(b-r)T}$ units, and borrowing Xe^{-rT} at the riskless rate of interest. The initial and terminal values of this portfolio are presented in Table 10.5. Note that the terminal values are equal to zero, independent of whether or not the terminal commodity price is above or below the exercise price of the options. If the call is in-the-money at expiration, we are required to deliver one unit of the commodity and receive X. By virtue of the rollover position, we have one unit of the commodity on hand to make the delivery. The exercise proceeds are used to cover the riskless borrowings, and the put option expires worthless. If the put is in-the-money at expiration, we exercise the put, delivering the commodity and receiving X. The remaining transactions

TABLE 10.5 Arbitrage transactions for establishing put-call parity for European options.

$$c(S,T;X) - p(S,T;X) = Se^{(b-r)T} - Xe^{-rT}$$

Position	Initial Value	Terminal Value $\tilde{S}_T \leq X$	Terminal Value $\tilde{S}_T > X$
Sell European call	c	0	$-(\tilde{S}_T - X)$
Buy European put	$-p$	$X - \tilde{S}_T$	0
Buy rollover position in commodity	$-Se^{(b-r)T}$	$\tilde{S}_T$	$\tilde{S}_T$
Borrow Xe^{-rT}	Xe^{-rT}	$-X$	$-X$
Net portfolio value	$c - p - Se^{(b-r)T} + Xe^{-rT}$	0	0

are as described above. Since the terminal values are certain to be equal to zero, it must be the case that no one would pay a price other than zero to take on the portfolio. Setting the net initial portfolio value equal to zero produces equation (10.22).

The put-call parity relation for European futures options is a special case of (10.22), where the cost-of-carry rate, b, equals zero. (Recall that futures positions require no initial investment outlay.) The relation is

$$c(F,T;X) - p(F,T;X) = e^{-rT}(F - X). \tag{10.23}$$

This relation first appeared in Black (1976).

American Options

The early exercise feature of American options causes the specification of the put-call parity relation to be different from that for European options. The relations linking the commodity price and the American commodity options prices,

$$S - X \leq C(S,T;X) - P(S,T;X) \leq Se^{(b-r)T} - Xe^{-rT}, \text{ if } b \geq r, \tag{10.24a}$$

and

$$Se^{(b-r)T} - X \leq C(S,T;X) - P(S,T;X) \leq S - Xe^{-rT}, \text{ if } b < r, \tag{10.24b}$$

must be developed through two separate sets of arbitrage transactions. We consider each inequality in turn.

The left-hand side condition of (10.24a) can be derived by considering the values of a portfolio that consists of buying a call, selling a put, lending X risklessly, and selling a rollover position in the commodity starting with one unit and decreasing the position by the factor $e^{-(b-r)}$ each day, and lending X risklessly. Table 10.6 contains these portfolio values. Note that there is now an additional column in the table with the heading "Intermediate Value." Because the portfolio holder is short an option, she runs the risk of being assigned delivery on the option prior to expiration. We must account for this possibility in deriving rational price bounds for American options.

In Table 10.6, it can be seen that, if all of the security positions stay open until expiration, the terminal value of the portfolio will be positive, independent of whether the terminal commodity price is above or below the exercise price of the options. If the terminal commodity price is above the exercise price, the call option is exercised, and the commodity acquired at exercise price X is used to deliver, in part, against the short commodity position. If the terminal commodity price is below the exercise price, the put option holder exercises her option by selling us the underlying commodity at exercise price X. In turn, we use the commodity to deliver against our short commodity position established at the outset. Therefore, if the option positions are held to expiration, the portfolio terminal value is certain to be positive.

In the event the put option holder decides to exercise her option early at time t, the investment in the riskless bonds is more than sufficient to cover the payment

TABLE 10.6 Arbitrage transactions for establishing put-call parity for American options, where $b \geq r$.

$$S - X \leq C(S,T;X) - P(S,T;X)$$

Position	Initial Value	Put Exercised Early: Intermediate Value	Put Exercised at Expiration: Terminal Value $\tilde{S}_T \leq X$	Put Exercised at Expiration: Terminal Value $\tilde{S}_T > X$
Buy American call	$-C$	$\tilde{C}_t$	0	$\tilde{S}_T - X$
Sell American put	P	$-(X - \tilde{S}_t)$	$-(X - \tilde{S}_T)$	0
Sell rollover position in commodity	S	$-\tilde{S}_t e^{-(b-r)t}$	$-\tilde{S}_T e^{-(b-r)T}$	$-\tilde{S}_T e^{-(b-r)T}$
Lend X	$-X$	Xe^{rt}	Xe^{rT}	Xe^{rT}
Net portfolio value	$-C + P + S - X$	$\tilde{C}_t + X(e^{rt} - 1) + \tilde{S}_t[1 - e^{-(b-r)t}]$	$X(e^{rT} - 1) + \tilde{S}_T[1 - e^{-(b-r)T}]$	$X(e^{rT} - 1) + \tilde{S}_T[1 - e^{-(b-r)T}]$

of the exercise price to the put option holder, and the commodity received from the exercise of the put is used to cover the commodity sold when the portfolio was formed. In addition, we still hold a call option, which may have significant value. In other words, by forming the portfolio of securities in the proportions noted above, we have formed a portfolio that will never have a negative future value. If the future value is assured to be nonnegative, the initial value is assured to be nonpositive, or

$$-C(S,T;X) + P(S,T;X) + S - X \leq 0.$$

Rearranging provides the left-hand side of equation (10.24a). The left-hand side of (10.24b) can be established using arbitrage transactions and arguments similar to those in Table 10.6, except that the rollover commodity position begins with an investment of $e^{(b-r)T}$ units.

The right-hand side of (10.24a) may be derived by considering the portfolio used to prove European put-call parity. Changing the notation to reflect the fact that we are discussing American options and introducing the "Intermediate Value" column to reflect the prospect of early exercise, the portfolio value table becomes Table 10.7. Here, the terminal value of the portfolio is certain to be equal to zero, if the option positions stay open until that time. The option positions are offset by the commodity position and the riskless borrowings are offset by the exercise prices of the options. In the event the American call option holder decides to exercise his call option early, the portfolio holder uses her long commodity position to cover her commodity obligation on the exercised call and uses the exercise proceeds to retire her outstanding debt. After these actions are taken, she still has an open long put position, cash in the amount of $X[1 - e^{-r(T-t)}]$, and a commodity position worth $S[e^{(b-r)(T-t)} - 1]$. Since the portfolio is certain to have nonnegative outcomes, the initial value must be nonpositive or

$$C(S,T;X) - P(S,T;X) - Se^{(b-r)T} + Xe^{-rT} \leq 0. \quad \textbf{(10.25)}$$

Rearranging provides the right-hand side of the American put-call parity condition (10.24a). The right-hand side of (10.24b) can be established by considering the portfolio in Table 10.7, with the exception that the rollover commodity position begins with one unit instead of $e^{(b-r)T}$.

The put-call parity relation for American futures options is a special case of (10.24b). Since the cost-of-carry rate, b, equals zero, the relation becomes

$$Fe^{-rT} - X \leq C(F,T;X) - P(F,T;X) \leq F - Xe^{-rT}. \quad \textbf{(10.26)}$$

Table 10.8 contains a summary of the put-call parity relations developed in this section.

TABLE 10.7 Arbitrage transactions for establishing put-call parity for American options, where $b \geq r$.

$$C(S,T;X) - P(S,T;X) \leq Se^{(b-r)T} - Xe^{-rT}$$

		Call Exercised Early	Call Exercised at Expiration	
Position	Initial Value	Intermediate Value	Terminal Value $\tilde{S}_T \leq X$	Terminal Value $\tilde{S}_T > X$
Sell American call	C	$-(\tilde{S}_t - X)$	0	$-(\tilde{S}_T - X)$
Buy American put	$-P$	$\tilde{P}_t$	$X - \tilde{S}_T$	0
Buy rollover position in commodity	$-Se^{(b-r)T}$	$\tilde{S}_t e^{(b-r)(T-t)}$	$\tilde{S}_T$	$\tilde{S}_T$
Borrow Xe^{-rT}	Xe^{-rT}	$-Xe^{-r(T-t)}$	$-X$	$-X$
Net portfolio value	$C - P - Se^{(b-r)T}$ $+Xe^{-rT}$	$\tilde{P}_t + X[1 - e^{-r(T-t)}]$ $+S[e^{(b-r)(T-t)} - 1]$	0	0

TABLE 10.8 Summary of put-call parity relations for commodity and futures options.

Option Type	Commodity Options
European	$c(S,T;X) - p(S,T;X) = Se^{(b-r)T} - Xe^{-rT}$
American	$S - X \leq C(S,T;X) - P(S,T;X) \leq Se^{(b-r)T} - Xe^{-rT}$, if $b \geq r$
	$Se^{(b-r)T} - X \leq C(S,T;X) - P(S,T;X) \leq S - Xe^{-rT}$, if $b < r$
	Futures Options
European	$c(F,T;X) - p(F,T;X) = e^{-rT}(F - X)$
American	$Fe^{-rT} - X \leq C(F,T;X) - P(F,T;X) \leq F - Xe^{-rT}$

10.8 COMMODITY OPTIONS VERSUS FUTURES OPTIONS

Thus far in the chapter, we have discussed the arbitrage linkages between options and their underlying instruments, as depicted by the horizontal line segments in Figure 10.1. In this section, we complete the discussion by focusing on the arbitrage price relations that exist between commodity options and futures options, should both markets exist for a particular underlying commodity.[5]

European Options

The relation between commodity option and futures option prices is straightforward for European options. Since the futures contract has the same time to expiration as the option contracts (by assumption (e) at the beginning of the chapter), and since the price of the futures contract equals the underlying commodity price at expiration, the European call (put) option on a futures contract will have exactly the same value as the European call (put) option written on the underlying commodity itself. That is,

$$c(S, T; X) = c(F, T; X) \tag{10.27a}$$

and

$$p(S, T; X) = p(F, T; X). \tag{10.27b}$$

In the case of European options, commodity options and futures options are perfect substitutes for one another.

American Options

The equality of the European option prices arises because the options cannot be exercised early and at the options' expiration the futures price equals the commodity price. American options, however, have an early exercise privilege and must reflect the fact that, prior to expiration, the futures price may differ from the commodity price. When the futures price is at least as great as the price of the underlying commodity (i.e., in the cost-of-carry model, $F = Se^{bT}$, $F \geqslant S$ if $b \geqslant 0$), the American call option written on the futures is worth at least as much as the American call written on the commodity,

$$C(F, T; X) \geq C(S, T; X). \tag{10.28a}$$

[5]Recall that in Chapter 1 we noted that commodity options and futures options markets exist for many foreign currencies such as the German Deutsche mark and for stock indexes such as the S&P 500.

To see this, consider the initial, intermediate and terminal values of a portfolio that consists of a long position in the futures option and a short position in the commodity option, as illustrated in Table 10.9. If both options are held to expiration, the net terminal value of the portfolio equals zero. If the options are out-of-the-money they expire worthless, and if they are in-the-money the options' payoffs negate each other. In the event the American commodity option is exercised early against the portfolio holder, the value of the portfolio is $\tilde{C}_t - \tilde{S}_t + X$. But, the lower price bound of the call is $\tilde{F}_t - X$. Since we have assumed $F_t \geqslant S_t$, the intermediate value of the portfolio is nonnegative. In the absence of costless arbitrage opportunities, the initial value of the portfolio must be nonpositive, so condition (10.28a) must hold.

A similar arbitrage argument can be developed for American put options. Since a put option represents the right to sell the underlying instrument, the instrument with the lowest price provides the highest option value. Thus, if $F \geqslant S$,

$$P(S,T;X) \geq P(F,T;X). \qquad \textbf{(10.28b)}$$

Conditions (10.28a) and (10.28b) present the price relations between American options written on commodities and futures in the usual case where the cost-of-carry rate, b, is positive or equal to zero. In a few markets, however, b may be less than zero. For example, with foreign currencies, $b < 0$ when the foreign riskless rate of interest is greater than the domestic riskless rate (as shown in Chapter 9). When this happens, it can be easily shown, using arguments similar to those above, that the conditions (10.28a) and (10.28b) will be reversed. Table 10.10 provides a summary of the price relations developed in this section.

TABLE 10.9 Arbitrage transactions for establishing price relation between commodity and futures options, where $b \geq 0$.

$$C(F,T;X) \geq C(S,T;X)$$

		Call Exercised Early	Call Exercised at Expiration	
Position	Initial Value	Intermediate Value	Terminal Value $\tilde{S}_T \leq X$	$\tilde{S}_T > X$
Buy futures option	$-C(F,T;X)$	$C(\tilde{F}_t, T-t;X)$	0	$\tilde{S}_T - X$
Sell commodity option	$C(S,T;X)$	$-(\tilde{S}_t - X)$	0	$-(\tilde{S}_T - X)$
Net portfolio value	$C(S,T;X)$ $-C(F,T;X)$	$C(\tilde{F}_t, T-t;X)$ $-\tilde{S}_t + X$	0 0	0 0

TABLE 10.10 Summary of price relations between commodity and futures options.

Option Type	Call Option	Put Option
European	$c(S,T;X) = c(F,T;X)$	$p(S,T;X) = p(F,T;X)$
American	$C(S,T;X) \leq C(F,T;X)$ if $b \geq 0$ $C(S,T;X) > C(F,T;X)$ if $b < 0$	$P(S,T;X) \geq P(F,T;X)$ if $b \geq 0$ $P(S,T;X) < P(F,T;X)$ if $b < 0$

10.9 SUMMARY

We show first how option positions can be characterized using vector notation. Lower bounds on the prices of European and American options are derived and explained.

The distinction between European and American options is the early exercise privilege of American options. We discuss the conditions under which the early exercise privilege has value. First, early exercise will only occur if the option is substantially in the money. Second, for call options, early exercise requires that the cost of carrying the commodity is "small" relative to the interest rate; and, for put options, early exercise requires that the cost of carrying the commodity is "high" relative to the interest rate.

The put-call parity relation, linking the prices of options and the underlying commodity, is derived for European and American options and for commodity options and futures options. Finally, the relation between the price of a commodity option and a futures option is derived.