# 7 STOCK INDEX FUTURES CONTRACTS

Arguably, the most exciting financial innovation of the 1980s has been the introduction of stock index futures contracts. These contracts, written on the value of various stock index portfolios, provide important benefits to stock portfolio managers. The uses and benefits of these contracts are described in this chapter. We begin with a description of the history of stock index futures contracts in the U.S. and an explanation of current contract designs. The second section details the composition of the stock indexes that underlie currently traded index futures contracts. Section 3 describes the index arbitrage that holds the cost of carry relation in alignment and explains the concept of "program trading." In section 4, the intraday price behavior of the index and its futures contracts is investigated to see how well the price movements in the two markets are synchronized. The chapter concludes with an illustration of hedging with stock index futures contracts.

## 7.1 STOCK INDEX FUTURES MARKETS

The first stock index futures contract was introduced in February 1982 by the Kansas City Board of Trade. This contract, the Value Line futures contract, is written on the Value Line Composite Index, a stock index that consists of approximately 1700 stocks from the New York, American, and OTC stock markets.[1] The Chicago Mercantile Exchange quickly followed suit in April 1982 with a futures contract on the S&P 500 stock index, and then the Chicago Board of Trade in July 1984 followed with a futures contract on the Major Market Index. Other stock index

[1] The composition of the various stock indexes is discussed in the next section.

futures on over-the-counter stocks have been introduced, but most have failed. Table 7.1 contains the contract specifications of the five stock index futures contracts currently active in the U.S.

By far the most active stock index futures contract is that on the S&P 500 index. Table 7.1 shows that this contract trades at the Chicago Mercantile Exchange from 8:30 AM to 3:15 PM (CST). On a given day, S&P 500 futures contracts extending out four different maturities may trade. The contract maturities will be the following March, June, September, and December. The last trading day of the S&P 500 futures contract is the third Thursday of the contract month. Cash settlement of the contract takes place at the opening prices of the index stocks on Friday.[2] The contract denomination is 500 times the futures price. On November 13, 1991, for example, the December 1991 futures price was $398.30, so the stock equivalent of the futures is $398.30 × 500 or $199,150. The minimum price increment for changes in the futures price is $.05 × 500 or $25. As of April 1991, the initial speculative margin for the S&P 500 contracts was $22,000,[3] and the maintenance margin was $9,000.

The specifications of the other index contracts are also shown in Table 7.1. Next to the S&P 500, the most active markets are for the futures contracts on the NYSE Composite Index and the Major Market Index. The Value Line futures contracts have never been particularly active relative to their counterparts on the other futures exchanges, probably because of the way in which the index level is computed.[4] The only difference between the Value Line and Mini Value Line index futures contracts is that the latter contract is one-fifth the size of the former.

Table 7.2 contains a clipping from the *Wall Street Journal* showing prices for the various index futures contracts as of the close of trading on Wednesday, November 13, 1991. Only the three nearby S&P 500 futures contracts were active on November 13—the December 1991 and the March and June 1992 contracts. The estimated trading volume on that day was 42,125 contracts. The implied dollar stock equivalent of this volume of trading is at least $398.30 × 500 × 42,125 or $8.39 billion. As is usually the case, the nearby futures contract is the most active, as is reflected through the higher open interest figure for the December contract. The underlying S&P 500 index level, 397.42, is also reported in the table, just below the futures price summary.

## 7.2 COMPOSITION OF STOCK INDEXES

The indexes underlying the futures contracts contained in Tables 7.1 and 7.2 fall into one of three general categories: (a) value-weighted arithmetic stock indexes;

[2]In June 1987, the Chicago Mercantile Exchange and the New York Futures Exchange changed the settlement of their S&P 500 and NYSE index futures contracts from the close of trading to the open in an attempt to mitigate concern about occasional abnormal stock price movements in the "triple witching hour." The futures contracts on the Major Market and Value Line indexes continue to settle at the close. For an analysis of the effects of this change, see Stoll and Whaley (1991).

[3]Margins are adjusted when the risk of the underlying index changes perceptibly. Prior to the October 19, 1987, stock market crash, speculative margin on the S&P 500 futures contract was $6,000. Immediately following the crash, speculative margins were set as high as $20,000.

[4]The index composition is described later in this chapter.

TABLE 7.1 Contract specifications of stock index futures contracts trading in the U.S.

| Index (Exchange) | Trading Hours | Contract Months[a] | Units/ Minimum Price Fluctuation | Last Day of Trading[b] |
|---|---|---|---|---|
| S&P 500 (CME) | 8:30–3:15 (CST) | 3,6,9,12 | 500 x index/ .05 ($25) | Third Thursday |
| NYSE Index (NYFE) | 9:30–4:15 (EST) | 3,6,9,12 | 500 x index/ .05 ($25) | Thursday preceding third Friday |
| Major Market Index (CBOT) | 8:30–3:15 (CST) | 3 current months plus 3,6,9,12 | 250 x index/ .05 ($12.50) | First business day prior to Saturday following third Friday |
| Value Line Index (KC) | 8:30–3:15 (CST) | 3,6,9,12 | 500 x index/ .05 ($25) | Third Friday |
| Mini Value Line Index (KC) | 8:30–3:15 (CST) | 3,6,9,12 | 100 x index/ .05 ($5) | Third Friday |

*a.* The notation used in this column corresponds to the month of the calendar year (e.g., 1 is January, 2 is February, and so on).

*b.* All stock index futures contracts are cash settled.

(b) price-weighted arithmetic indexes; and (c) equal-weighted geometric indexes. The term *arithmetic* refers to the fact that the market values or returns of the individual stocks are "added up." The term *geometric* refers to the case where the values or returns are "multiplied." The S&P 500 and NYSE Composite indexes are in the first category; the Major Market Index falls in the second; and the Value Line Index falls in the third.

## Value-Weighted Arithmetic Indexes

The "value" of the common stocks in a value-weighted index refers to the total market capitalization of the firm's outstanding shares, that is, the number of shares outstanding ($n_{i,t}$) times the current price per share ($p_{i,t}$). The total market value of the index at time $t$ is therefore

$$\text{Total market value of index}_t = \sum_{i=1}^{N} n_{i,t} p_{i,t}, \tag{7.1}$$

TABLE 7.2 Stock index futures contract prices at the close of trading on Wednesday, November 13, 1991.

## FUTURES

**S&P 500 INDEX (CME) 500 times index**

| | Open | High | Low | Settle | Chg | High | Low | Open Interest |
|---|---|---|---|---|---|---|---|---|
| Dec | 395.00 | 398.50 | 394.30 | 398.30 | + 1.00 | 401.50 | 316.50 | 139,341 |
| Mr92 | 396.80 | 400.50 | 396.50 | 400.35 | + 1.00 | 404.00 | 374.70 | 7,544 |
| June | 398.30 | 402.35 | 398.30 | 402.20 | + 1.10 | 407.00 | 379.00 | 1,102 |

Est vol 42,125; vol Tues 41,413; open int 148,048, +916.
Indx prelim High 397.42; Low 394.01; Close 397.42 +.68

**NIKKEI 225 Stock Average (CME)–$5 times NSA**

| | Open | High | Low | Settle | Chg | High | Low | Open Interest |
|---|---|---|---|---|---|---|---|---|
| Dec | 24690. | 24700. | 24600. | 24700. | – 340. | 28900. | 22380. | 10,869 |
| Mr92 | 25250. | 25250. | 25170. | 25230. | – 340. | 26725. | 23000. | 2,423 |

Est vol 1,107; vol Tues 1,132; open int 13,292, +467.
The index: High 24814.35; Low 24416.23; Close 24416.23 – 251.50

**NYSE COMPOSITE INDEX (NYFE) 500 times index**

| | Open | High | Low | Settle | Chg | High | Low | Open Interest |
|---|---|---|---|---|---|---|---|---|
| Dec | 218.00 | 220.10 | 217.75 | 220.05 | + .70 | 220.10 | 175.50 | 5,026 |
| Mr92 | 218.80 | 221.00 | 218.80 | 221.00 | + .80 | 221.00 | 207.60 | 746 |
| June | .... | .... | .... | 222.00 | + .80 | 220.10 | 208.90 | 172 |
| Sept | .... | .... | .... | 223.00 | + .80 | 221.00 | 217.50 | 123 |

Est vol 5,057; vol Tues 5,996; open int 6,067, +344.
The index: High 219.37; Low 217.64; Close 219.37 +.37

**MAJOR MKT INDEX (CBT) $500 times index**

| | Open | High | Low | Settle | Chg | High | Low | Open Interest |
|---|---|---|---|---|---|---|---|---|
| Nov | 323.70 | 327.40 | 323.25 | 327.25 | + 1.55 | 327.40 | 315.20 | 2,819 |
| Dec | 323.50 | 327.70 | 323.50 | 327.70 | + 1.50 | 327.70 | 315.75 | 746 |

Est vol 2,500; vol Tues 1,163; open int 3,598, +122.
The index: High 327.25; Low 323.58; Close 327.25 +1.28

**MGMI BASE METAL INDEX (FOX) 100 times index**

| | Open | High | Low | Settle | Chg | High | Low | Open Interest |
|---|---|---|---|---|---|---|---|---|
| Nov | .... | .... | .... | 134.50 | .... | 140.50 | 132.50 | 2,246 |
| Dec | .... | .... | .... | 134.80 | .... | 182.70 | 133.00 | 8,662 |
| Ja92 | .... | .... | .... | 134.90 | .... | 137.10 | 132.30 | 120 |
| Mar | .... | .... | .... | 135.30 | .... | 160.20 | 132.60 | 2,643 |
| June | .... | .... | .... | 136.00 | .... | 155.90 | 134.50 | 962 |
| Sept | .... | .... | .... | 136.80 | .... | 146.60 | 134.60 | 149 |

Est vol 0; vol Tues 0; open int 14,712, .
The index: High 134.52; Low 133.58; Close 134.03 +.81

## OTHER FUTURES

Settlement price of selected contract. Volume and open interest of all contract months.

**KC Mini Value Line (KC)–100 times index**
Dec 328.60 +.85; Est. vol. 100; Open int. 254
**KC Value Line Index (KC)–500 times index**
Dec 328.30 +.70; Est. vol. 250; Open int. 1,722
The index: High 326.47; Low 324.48; Close 326.47 +.24
**CRB Index (NYFE)–500 times index**
Dec 214.90 +.35; Est. vol. 206; Open int. 1,221
The index: High 214.43; Low 213.94; Close 214.20 +.26

CBT–Chicago Board of Trade. CME–Chicago Mercantile Exchange. KC–Kansas City Board of Trade. NYFE–New York Futures Exchange, a unit of the New York Stock Exchange.

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where $N$ is the number of stocks in the index. This market value is then scaled by a divisor so that the index in period $t$ is

$$S_t = \frac{\sum_{i=1}^{N} n_{i,t} p_{i,t}}{\text{Divisor}_t}. \tag{7.2}$$

The divisor represents what the stocks currently in the index would have been worth in a base period. In the base period, the divisor is the market value of the stocks in the index,

$$\text{Divisor}_0 = \sum_{i=1}^{N} n_{i,0} p_{i,0}. \tag{7.3}$$

Over time, the numerator of (7.2) changes because stocks enter or leave the index or because shares are issued or repurchased by companies. Because such changes do not reflect a change in the value of the stocks, an adjustment to the divisor is made on the day that a change in the index composition occurs. The new divisor on day $t$ is just the old divisor on day $t$ adjusted by the ratio of the market value of the new index composition on day $t$ divided by the market value of the old index composition on day $t$,

$$\text{new divisor}_t = \left( \frac{\text{market value new}_t}{\text{market value old}_t} \right) \text{old divisor}_t. \tag{7.4}$$

Both the S&P 500 and NYSE Composite indexes are value-weighted. The S&P 500 consists of 500 common stocks, the majority of which trade on the NYSE, although about fifty stocks trade on the American Exchange and in the OTC market. The index was designed by Standard & Poors' to contain stocks from a broad variety of industry groupings. The market value for the base period of the S&P 500 is based on the average market values of the component stocks during the years 1941 through 1943. At that time, the index was set equal to 10. The NYSE Composite contains all common stocks traded on the NYSE, slightly more than 1,500 in number. The base period for the NYSE index is December 31, 1965, at which time the index was set equal to 50. As Table 7.2 shows, the values of the S&P 500 and NYSE Composite stocks indexes were 397.42 and 219.37, respectively, at the close of trading on November 13, 1991, reflecting percentage gains of 3,874 percent and 339 percent, respectively, from their base periods.

## Price-Weighted Arithmetic Indexes

A price-weighted arithmetic index is like a value-weighted arithmetic index, except that the number of shares outstanding does not play a role. The price-weighted arithmetic index is computed as

$$S_t = \frac{\sum_{i=1}^{N} p_{i,t}}{\text{Divisor}_t}. \tag{7.5}$$

In a price-weighted index, the divisor in the base period equals the sum of the prices of the stocks in the base period, that is,

$$\text{Divisor}_0 = \sum_{i=1}^{N} p_{i,0}. \tag{7.6}$$

Like a value-weighted index, the divisor of a price-weighted index is adjusted to reflect stock splits and stock dividends so that the index level remains unchanged during the stock split/stock dividend process [i.e., in the manner of (7.4)]. Unlike the value-weighted index, however, the divisor of the price-weighted index is unaffected by new stock issues or share repurchases.

The best known price-weighted arithmetic index is the Dow Jones Industrial Average (DJIA), which consists of thirty "blue-chip" stocks. In an attempt to create an index that mimics the price movements of the DJIA, the American Exchange created the Major Market Index (MMI). This price-weighted index contains twenty stocks, seventeen of which are also members of the DJIA. Table 7.2 shows that the value of the MMI at the close of trading on November 13, 1991, was 327.25.

## Equal-Weighted Geometric Indexes

An equal-weighted geometric index is somewhat peculiar. To compute it, a geometric average of the rates of return of the individual stocks within the index over a period ($R_{i,t}$) is taken, that is,

$$R_{S,t} = \sqrt[N]{\prod_{i=1}^{N}(1 + R_{i,t})} - 1. \tag{7.7}$$

This return is used to update the index from the previous period,

$$S_t = S_{t-1}(1 + R_{S,t}). \tag{7.8}$$

Currently, the only equal-weighted geometric index is the Value Line Index. It consists of approximately 1,700 stocks. Approximately ninety percent of the Value Line index capitalization is from shares traded on the NYSE, one percent from AMEX, and nine percent OTC. The Value Line index and its futures contracts are of limited interest for two reasons. First, the index weights all stocks equally so small stocks have as much impact on the index movements as large stocks. For an index to track the behavior of the "market," much greater weight should be placed on large capitalization issues. Second, geometric averaging causes the rate of return on the index to be less than the rate of return that would be earned by an equal-weighted investment in each of the 1,700 stocks. As a result, price movements (returns) of the Value Line index are not as strongly correlated with most stock portfolios as are other indexes, which makes the Value Line futures contract less useful for hedging purposes. Table 7.2 shows that the open interest of the Value Line futures is much lower than the futures contracts on the other indexes. The Value Line index closed at 326.47 on November 13, 1991.

## Stock Index Simulations

The arithmetic versus geometric averaging of the various stock indexes warrants further discussion, and the discussion is best facilitated through a numerical example. Assume that there are two stocks, $A$ and $B$, in the marketplace. Both are priced at \$20 per share, and both have 100 shares outstanding. Neither stock pays dividends. Table 7.3 shows sample paths for the prices of each stock over a twelve-month period. Alongside of the stock prices are: (a) a value-weighted arithmetic index, (b) a price-weighted arithmetic index, and (c) an equal-weighted geometric index corresponding to these two stocks. All the indexes are created to have a base value of 100 at time 0. The index values are computed using equations (7.2), (7.5), and (7.8), respectively.

In Table 7.3, note two things. First, the value-weighted and price-weighted arithmetic indexes have identical values. This is because the simulation begins with equal investments in both stocks (the stocks' market capitalizations and prices per share are equal). The price movements of these indexes are perfectly positively correlated with any equal-weighted portfolio of these two common stocks formed at time 0. Second, the equal-weighted geometric index has a terminal value considerably below the terminal values of the other two indexes, 134.16 versus 140.00. This is the downward bias discussed earlier. The price movements of a geometric index in general do not correspond to price movements in a stock portfolio, so

TABLE 7.3 Simulation of value-weighted arithmetic, price-weighted arithmetic, and equal-weighted geometric stock index values created using two stocks.

| Time $t$ | Stock $A$ | Stock $B$ | Value-Weighted Arithmetic Index[a] | Price-Weighted Arithmetic Index[b] | Equal-Weighted Geometric Index[c] |
|---|---|---|---|---|---|
| 0 | 20 | 20 | 100.00 | 100.00 | 100.00 |
| 1 | 25 | 16 | 102.50 | 102.50 | 100.00 |
| 2 | 30 | 20 | 125.00 | 125.00 | 122.47 |
| 3 | 33 | 22 | 137.50 | 137.50 | 134.72 |
| 4 | 27 | 20 | 117.50 | 117.50 | 116.19 |
| 5 | 36 | 15 | 127.50 | 127.50 | 116.19 |
| 6 | 40 | 16 | 140.00 | 140.00 | 126.49 |
| 7 | 36 | 18 | 135.00 | 135.00 | 127.28 |
| 8 | 38 | 21 | 147.50 | 147.50 | 141.24 |
| 9 | 40 | 18 | 145.00 | 145.00 | 134.16 |
| 10 | 38 | 21 | 147.50 | 147.50 | 141.24 |
| 11 | 40 | 22 | 155.00 | 155.00 | 148.32 |
| 12 | 36 | 20 | 140.00 | 140.00 | 134.16 |

*a*. The value-weighted arithmetic index consists of 100 shares of Stock $A$ and 100 shares of Stock $B$. At time 0, the market capitalization is 4,000, which is adjusted to an index level of 100.

*b*. The price-weighted arithmetic index at time 0 equals the sum of the share prices of Stock $A$ and Stock $B$ divided by the divisor.

*c*. The equal-weighted geometric index equals 100 in the base period. The value at time 1 equals the time 0 index value times the square root of the product of one plus the rate of return on Stock $A$ and one plus the rate of return on Stock $B$.

futures contracts on a geometric index are of less value for hedging purposes than are futures contracts on an arithmetic index.

## Correlation Among Index Returns

Still more intuition about the different stock indexes can be gathered by examining actual weekly rates of price appreciation in selected U.S. stock indexes. Table 7.4 contains the means and standard deviations of the percentage rates of price appreciation of six different stock indexes. Also included in the table are estimated contemporaneous correlation coefficients between each pair of return series. Weekly returns are computed using closing index levels each Wednesday during the calendar year 1989. Several interesting results appear in the table.

First, note that the standard deviation of the rate of return for the arithmetic indexes is highest for MMI—1.7453 percent per week. This result is not surprising

TABLE 7.4 Summary statistics of weekly percentage rates of price appreciation in five U.S. stock indexes during the calendar year 1989.[a]

Means and Standard Deviations of Index Returns

| Index | Mean Return | Standard Deviation |
|---|---|---|
| DJIA | 0.4537 | 1.6640 |
| MMI | 0.5072 | 1.7453 |
| S&P 500 | 0.4481 | 1.5825 |
| VL | 0.1809 | 1.3180 |
| NYSE | 0.4127 | 1.4916 |

Contemporaneous Correlations Between Pairs of Index Returns

| Index | MMI | S&P 500 | VL | NYSE |
|---|---|---|---|---|
| DJIA | .9779 | .9774 | .8880 | .9750 |
| MMI | | .9497 | .8104 | .9403 |
| S&P 500 | | | .9137 | .9972 |
| VL | | | | .9337 |

*a*. Rates of price appreciation are computed on the basis of the closing index levels each Wednesday during 1989. Cash dividends paid on index stocks are not considered.

considering that the MMI has the fewest stocks of any of the indexes examined. The reduction in standard deviation from the MMI to the DJIA to the S&P 500, and, finally, to the NYSE reflects increasingly higher degrees of diversification. The DJIA has 30 stocks, the S&P 500 has 500, and the NYSE has more than 1500. The standard deviation of the return of the Value Line index reflects both diversification and a downward bias due to the way in which the index is computed. (Recall the geometric averaging discussed earlier in this section.)

Second, note that the correlation between pairs of return series is highest for the S&P 500 and the NYSE indexes—0.9972. Both of these indexes are value-weighted and are highly diversified. The rates of return of the two stock indexes are virtually perfectly positively correlated.

Third, the returns of the MMI and the DJIA are also strongly positively correlated—0.9779. One would expect this to be the case given that seventeen of the stocks in the MMI are also in the DJIA. The fact that these indexes are not well-diversified, however, attenuates to a small degree the correlation between the returns of these two indexes.

Finally, while the correlation among the returns of any pair of arithmetic indexes is very high (approximately 0.93 or higher), the correlation between the returns of the Value Line index and any of the other indexes is relatively much lower. The geometric averaging of the returns of the stocks in the Value Line index portfolio and the inclusion of many small companies undermines the index's co-movements with other indexes.

## 7.3 INDEX ARBITRAGE AND PROGRAM TRADING

The cost of carry relation (3.6) from Chapter 3 applies to the relation between the stock index futures price and the price of the underlying index under the assumption that the dividend yield rate $d$ is a constant, continuous proportion of the index price level. Active stock index arbitrage ensures that

$$F_t = S_t e^{(r-d)(T-t)}, \tag{7.9}$$

where $F_t$ and $S_t$ are the time $t$ prices of the futures contract and the underlying stock index, respectively. Note that the derivation of this relation in Chapter 3, as it applies to stock index arbitrage, implies that the cash dividends, as they accrue through time, are being reinvested in the stock index portfolio.

Assuming that cash dividends are a constant, continuous proportion of the index level may be inappropriate, particularly for a narrow-based index like the MMI, where the small number of stocks in the index portfolio implies an obvious discreteness and seasonality of cash dividend payments.[5] In such a case, an assumption that the amount $D_i$ and the timing $t_i$ of the discrete cash dividends paid during the futures contract life (i.e., between time $t$ and time $T$) are known is usually used. Furthermore, rather than assuming that the dividends are being reinvested in the stock index portfolio, dividends are assumed to be reinvested at the riskless rate of interest until the futures contract expires.

Under these assumptions, stock index arbitrage involves the transactions shown in Table 7.5a. The long position in the index portfolio provides a terminal value equal to the uncertain index price $\tilde{S}_T$ plus a known aggregate dividend income (plus accrued interest) $\sum_{i=1}^{n} D_i e^{r(T-t_i)}$. The stock portfolio position is financed completely with riskless borrowings, which are repaid at time $T$ at a cost $S_t e^{r(T-t)}$. The short futures position has a terminal value $-(\tilde{S}_T - F_t)$. Since the arbitrage portfolio involves a zero investment outlay and has no risk, the net terminal value of the portfolio must equal zero for the market to be in equilibrium. Thus, under the assumption of known discrete dividends, the cost-of-carry relation is

$$F_t = S_t e^{r(T-t)} - \sum_{i=1}^{n} D_i e^{r(T-t_i)}. \tag{7.10}$$

[5] Harvey and Whaley (1992) show pronounced seasonality in the cash dividends of the S&P 100 index, which contains approximately forty percent of the market value of the S&P 500 index. In particular, during the period 1983 through 1989, dividends tend to be highest in the months of February, May, August and November.

TABLE 7.5a Index arbitrage transactions for establishing the relation between index futures and underlying index prices, assuming known discrete cash dividends.

$$F_t = S_t e^{r(T-t)} - \sum_{i=1}^{n} D_i e^{r(T-t_i)}$$

| Position | Initial Value | Terminal Value |
|---|---|---|
| Buy index portfolio | $-S_t$ | $\tilde{S}_T + \sum_{i=1}^{n} D_i e^{r(T-t_i)}$ |
| Borrow $S_t$ | $S_t$ | $-S_t e^{r(T-t)}$ |
| Sell futures contract | 0 | $-(\tilde{S}_T - F_t)$ |
| Net portfolio value | 0 | $F_t - S_t e^{r(T-t)} + \sum_{i=1}^{n} D_i e^{r(T-t_i)}$ |

A simple version of the cost-of-carry relation arises if one assumes dividends and interest are paid at the end of the period corresponding to the life of the futures contract. Table 7.5b presents arbitrage transactions for this case and shows that this simple cost-of-carry relation is

$$F_t = S_t(1 + r^* - d^*), \quad \textbf{(7.11)}$$

where $r^*$ is the rate of interest and $d^*$ is the dividend yield over the remaining life of the futures contract.

Violations of the cost-of-carry relation (7.9), (7.10), or (7.11) signal profitable index arbitrage opportunities. If, for example, the observed futures price is above the theoretical futures price as implied by the right-hand side of (7.9), (7.10), or (7.11), arbitrageurs sell futures and buy the underlying stocks, driving the price of the futures down and the prices of stocks up. The arbitrage becomes unprofitable when the futures price reflects the cost of carrying the underlying stocks, that is, the interest cost less the cash dividends.

Unlike typical basis arbitrage, the underlying commodity is a precisely weighted *portfolio* of common stocks, rather than a single asset. For example, engaging in index arbitrage with the S&P 500 index requires a mechanism for buying or selling quickly and simultaneously all 500 stocks in the S&P 500 index portfolio. Since the simultaneous purchase or sale of the stocks in a precisely weighted and timely fashion is beyond human capability, computers and computer programs are usually used to place transaction orders as well as to assist in the execution of those orders. For this reason, trading of portfolios of stocks is called *program trading*, although program trades can also be done by manually preparing order tickets for each stock. NYSE statistics define a program trade as any order for a portfolio of 15 or more stocks.

TABLE 7.5b Index arbitrage transactions for establishing the relation between index futures and underlying index prices, assuming dividends and interest are paid at maturity.

$$F_t = S_t(1 + r^* - d^*)$$

| Position | Initial Value | Terminal Value |
|---|---|---|
| Buy index portfolio | $-S_t$ | $\tilde{S}_T + d^* S_t$ |
| Borrow $S_t$ | $S_t$ | $-S_t(1 + r^*)$ |
| Sell futures contract | 0 | $-(\tilde{S}_T - F_t)$ |
| Net portfolio value | 0 | $F_t - S_t(1 + r^* - d^*)$ |

## Treasury Bill Substitute

Technically speaking, one thinks of "index arbitrage" as being conducted by professional index arbitrageurs who establish offsetting positions in the manner shown in Table 7.5a. However, deviations from the cost of carry relation also offer opportunities for investors, such as pension funds, to structure an investment with index futures that offers a higher return than an investment of equivalent risk in another market. For example, if the futures price is high relative to the cost-of-carry equilibrium, fund managers can generate a riskless investment with a rate of return higher than the return on a Treasury bill of a maturity comparable to the index futures by selling index futures and buying the index portfolio. Such a strategy is called a *Treasury bill substitute.*

To understand how this strategy works, suppose that the current S&P 500 index level is 348.60 and that the nearby S&P 500 futures contract has a price of 354.50 and a time to expiration of 73 days. Suppose also that the future value of the S&P 500 dividends over the next 73 days is $2.79 and that a 73-day Treasury bill will provide a rate of return of 1.6 percent over its life. Using (7.11), the implied riskless rate of interest, $r^*$, on a 73-day investment involving selling the index futures and buying the stock index portfolio is determined by solving

$$354.50 = 348.60(1 + r^*) - 2.79.$$

The interest rate from the Treasury bill substitute strategy, $r^*$, is 2.5 percent. In other words, a pension fund that might ordinarily invest $3,486,000 in T-bills to earn 1.6 percent over 73 days could invest the same amount of money in a Treasury bill substitute to earn 2.5 percent over 73 days. To do so, the $3,486,000 is invested in the index portfolio (i.e., 10,000 units of the index are purchased) and twenty index futures contracts are sold (recall each index futures is 500 times the index value). Over the 73-day period, the index portfolio will generate $27,900 in cash dividends, and the index level will appreciate by 5.90 relative to the futures (because the futures price and index level converge at the end of 73 days), for a total price appreciation of $59,000. The overall rate of return on the Treasury bill substitute position is (27,900 + 59,000)/3,486,000 or 2.5 percent.

## Stock Replacement

A second example of how index futures may be used to generate a higher return than an investment with equivalent risk is a *stock replacement strategy.* When the actual futures price is below the theoretical futures price, an arbitrageur enacts a short arbitrage—the short sale of stocks and the purchase of futures contracts. But stock portfolio managers, too, can profit from such an opportunity by selling their stock portfolios and using the proceeds to buy index futures and Treasury bills, that is, by engaging in stock replacement.

To illustrate a stock replacement strategy, consider the previous example in which the current S&P 500 index level is 348.60, the time to expiration of the nearby S&P 500 futures contract is 73 days, the rate of return on a 73-day T-bill over the next 73 days is 1.60 percent, and the future value of the cash dividends on the S&P 500 over the next 73 days is $2.79. However, this time, assume the nearby S&P 500 futures price is $350.25. On the basis of these figures, the theoretical futures price is

$$F = 348.60(1.0160) - 2.79 = 351.39.$$

Since the observed futures price, $350.25, is less than its theoretical value, a stock replacement strategy can be used to generate a rate of return that will exceed the rate of return on a direct investment in the S&P 500 index portfolio without assuming more risk. A portfolio manager with $50,000,000 in the S&P 500 index portfolio will have a portfolio value of

$$\begin{aligned}\tilde{V}_{\text{S\&P 500},T} &= \left(\frac{50,000,000}{348.60}\right)(\tilde{S}_T + 2.79)\\ &= 143,430.87\tilde{S}_T + 400,172\end{aligned}$$

in 73 days. On the other hand, if he liquidates his S&P 500 stock portfolio and buys T-bills and the nearby S&P 500 futures contract, the portfolio value for the stock replacement strategy (SRS) will be

$$\begin{aligned}\tilde{V}_{\text{SRS},T} &= \left(\frac{50,000,000}{348.60}\right)(\tilde{S}_T - 350.25) + 50,000,000(1.016)\\ &= 143,430.87\tilde{S}_T - 50,236,662 + 50,800,000\\ &= 143,430.87\tilde{S}_T + 563,338.\end{aligned}$$

Note that the stock replacement strategy is certain to have a terminal value $163,166 higher than the stock portfolio strategy. The fact that this incremental value is certain reflects the fact that, while each strategy's terminal value is uncertain, both strategies have equal risk. If the observed futures price is below its theoretical level, however, the stock replacement strategy will dominate.

## Practical Considerations in Index Arbitrage

In practice, there are several reasons why deviations from the cost of carry relation do not ensure that arbitrage profits can be earned. First, and most important, are the transaction costs involved in trading the underlying index stocks. These include the commissions and the market impact costs of buying stocks at the ask price or selling stocks at the bid price. Procedures for trading portfolios of stock have improved dramatically in recent years and frequently involve the use of the NYSE computer entry system, DOT (Designated Order Turnaround). Nevertheless, these costs can be substantial, particularly if a number of portfolio transactions are hitting the market at the same time. Stock index arbitrageurs estimate the total round-trip transaction costs to be on the order of 0.5 to 0.75 percent of the underlying portfolio value.[6]

Second, the dividends in the cost-of-carry relation are assumed to be known with certainty. In general, this assumption is reasonable since firms tend to pay regular, constant, or constantly-increasing quarterly dividends. Any uncertainty about the anticipated dividend payments on the underlying stocks, however, introduces uncertainty about the return of the index arbitrage and can therefore limit arbitrage somewhat.

Third, certain types of arbitrage may involve risk. In some cases, arbitragers do not trade all the underlying stocks in the index. Instead, they buy or sell a representative basket of stocks because of the difficulty and the cost associated with transacting, say, all 500 of the stocks in the S&P 500 index. If the representative basket fails to move exactly like the underlying index, the arbitrage is risky.

Fourth, certain rules and regulations can impede arbitrage. For example, "circuit breakers" are now used to suspend index futures trading when the DJIA moves by more than a pre-specified amount in a given trading day. On such days, apparent arbitrage opportunities may be only illusory in the sense that the futures leg of the arbitrage may not be executable. Another example of an instance where a rule impedes arbitrage is when the arbitrage requires stocks to be sold and futures to be purchased. Since the index portfolio must be sold short, the *short-sale rule* comes into play. Under the short-sale rule, a stock is required to uptick before it may be sold short. When an entire portfolio of stocks must be sold, the time delay in waiting for an uptick in each stock makes the short arbitrage difficult to implement, so the futures price may tend to be less than or equal to its theoretical value. It is worthwhile to note that stock sales conducted by portfolio managers using stock

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[6] See Stoll and Whaley (1987, p. 18).

replacement strategies, however, are not subject to the uptick rule, and this will tend to limit the amount by which the futures price will fall below its theoretical price.[7]

Fifth, arbitrage is sometimes limited by the lack of capital. Brokerage firms may be limited by net capital requirement rules and the availability of higher yielding alternative fund uses. Moreover, many institutional investors may not be authorized to engage in index arbitrage.

The efficacy of the index arbitrage process has been examined in a number of theoretical and empirical papers.[8] In general, these papers find that observed futures prices can deviate from the theoretical futures price specified by arbitrage conditions by more than normal transaction costs. This is particularly the case for deviations of the futures price below the theoretical price. Such deviations may be difficult to arbitrage though because of the short sale restrictions and because of the lack of a sufficient number of institutions willing to engage in stock replacement strategies.

## 7.4 INTRADAY BEHAVIOR OF RETURNS

In perfectly efficient and continuous futures and stock markets absent transaction costs, riskless arbitrage profit opportunities should not appear so the cost-of-carry relation (7.9),

$$F_t = S_t e^{(r-d)(T-t)},$$

should be satisfied at every instant $t$ during the futures contract life. If such is the case, the instantaneous rate of price appreciation in the stock index equals the net cost-of-carry of the stock portfolio plus the instantaneous relative price change of the futures contract. To see this, take the natural logarithm of (7.9) at time $t$ and at time $t - 1$:

$$\ln S_t = -(r-d)(T-t) + \ln F_t, \tag{7.12}$$

[7]In August 1990, the NYSE implemented a rule requiring a downtick on each stock in an index arbitrage program purchase if the DJIA rose by 50 points or more and an uptick on each stock in an index arbitrage program sale (short or from a long position) if the DJIA declined by 50 points. This rule is counter productive because it impedes index arbitrage.

[8]Cornell and French (1983), Figlewski (1984a), Gastineau and Madansky (1983), Modest and Sundaresan (1983), Peters (1985), Stoll and Whaley (1986b), MacKinlay and Ramaswamy (1988), Kleidon (1991), Kleidon and Whaley (1991), and Miller, Muthuswamy, and Whaley (1991) examine the arbitrage process and consider possible explanations for observed deviations from theoretical prices. Other papers, notably Garcia and Gould (1987), Gould (1988), and Brennan and Schwartz (1990), analyze strategies for trading on mispricing.

and

$$\ln S_{t-1} = -(r - d)(T - t + 1) + \ln F_{t-1}, \quad \textbf{(7.13)}$$

and then subtract (7.13) from (7.12),

$$R_{S,t} = (r - d) + R_{F,t}, \quad \textbf{(7.14)}$$

where $R_{S,t} = \ln(S_t/S_{t-1})$ and $R_{F,t} = \ln(F_t/F_{t-1})$.

Several implications follow from (7.14) under the assumptions that the short-term interest rate and the dividend yield rate of the stock index are constant and that the index futures and stock markets are efficient and continuous:

a. The expected rate of price appreciation on the stock index portfolio $E(\tilde{R}_{S,t})$ equals the net cost of carry $(r - d)$ plus the expected rate of return on the futures contract $E(\tilde{R}_{F,t})$.
b. The standard deviation of the rate of return on the futures contract equals the standard deviation of the rate of return of the underlying stock index.
c. The contemporaneous rates of return of the futures contract and the underlying stock portfolio are perfectly positively correlated.
d. The rates of return of the futures contract and of the underlying stock index portfolio are serially uncorrelated.[9]
e. The noncontemporaneous rates of return of the futures contract and the underlying stock portfolio are uncorrelated.

Naturally, all of the above implications are based on the assumption that the cost-of-carry relation (7.9) holds at all points in time. It has been shown, however, that (7.9) does not hold exactly; indeed, one of the puzzles in stock index futures is the frequency with which deviations from the cost-of-carry relation are observed. Stoll and Whaley (1986b, Table 23A), for example, report frequent violations of the cost-of-carry relation in excess of transaction costs using hourly S&P 500 index and index futures data during the period April 1982 through December 1985. The frequency of violation is nearly eighty percent for the June 1982 futures contract. For more recent contract maturities, however, the frequency falls below fifteen percent. MacKinlay and Ramaswamy (1988, Table 6) report similar results for the S&P 500 futures contracts expiring in September 1983 through June 1987. Using fifteen-minute price data, they find that the cost-of-carry relation is violated 14.4 percent of the time on average.

Violations of the cost-of-carry relation may appear for a variety of reasons. Some, like transaction costs, were discussed in the last section. The presence of

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[9]Technically speaking, more than an assumption of market efficiency is needed to ensure serially uncorrelated rates of return. It must also be the case that the expected rates of return of the futures and stock index are constant. [See Fama (1976, pp. 149–151).] Such an assumption is reasonable since the rate of return series that we will examine below are intraday.

transaction costs tends to introduce noise in the rate of return relation (7.14). An important reason not mentioned in the last section is the infrequent trading of stocks within the index. Markets for individual stocks are not perfectly continuous. Consequently, stock index prices, which are averages of the last transaction prices of component stocks, lag actual developments in the stock market. Fisher (1966) describes this phenomenon. Cohen, *et al.* (1986, Ch. 6) give a more general discussion of serial correlation of stock index returns in terms of delays in the price adjustment of securities. Lo and MacKinlay (1988) model the effects of infrequent trading on index returns under certain restrictive assumptions. Assuming that the index futures prices instantaneously reflect new information, observed futures returns should be expected to lead observed stock index returns because of infrequent trading, even though there is no economic significance to this behavior whatsoever.

Stoll and Whaley (1990b) use five-minute, intraday rate of return data for the S&P 500 index and the nearby S&P 500 futures contracts to (a) model and purge the effects of infrequent trading in the stock index portfolio, and (b) assess the degree of simultaneity between returns in the index futures and stock markets. The effects of infrequent trading are shown in Table 7.6. Note that, while the S&P 500 futures contract returns have virtually no serial correlation, the returns of the S&P 500 index portfolio are strongly positively serially correlated. The first-order serial correlation in the S&P 500 index returns exceeds 0.5. Because not all stocks within the S&P 500 index portfolio trade in every five-minute interval, a market movement within this interval may not be recorded in the price of less actively traded stocks until some time later when the stock finally trades. The effect of this phenomenon is positive serial correlation in the portfolio return series. The serial correlation does not disappear until lag 4 or 5 using five-minute returns.

The effects of infrequent trading on observed stock index returns are modeled theoretically and estimated empirically in Stoll and Whaley (1990b). The residuals (return innovations) from the estimated model are examined to assess the degree of any remaining positive serial correlation. The last pair of columns in Table 7.6 show these results. With the effects of infrequent trading modeled and purged, the return innovations of the S&P 500 index are virtually white noise. None of the estimated serial correlation coefficients exceed 0.02 in absolute magnitude.

Finally, to assess the degree of simultaneity between the S&P 500 index futures and stock market returns, the return innovations of the S&P 500 index are regressed on lag, contemporaneous, and lead futures returns,

$$\epsilon_{S,t} = \alpha + \sum_{k=-3}^{3} \beta_k R_{F,t-k} + u_t. \tag{7.15}$$

The regression results are shown in Table 7.7. In addition, for purposes of comparison, the regression results of observed S&P 500 index returns regressed on lag, contemporaneous, and lead futures returns are also reported.

The return innovation regression results in Table 7.7 indicate that the dominant relation between the two markets is contemporaneous. The estimated coefficient of the contemporaneous futures return, $\hat{\beta}_0$, in the return innovation regression is 0.1338, higher than any of the leading or lagged coefficients. The estimated coef-

TABLE 7.6 Estimated serial correlation coefficients of observed returns of the S&P 500 index ($R^o_S$) and the S&P 500 index futures contract ($R^o_F$) for the 1249-day period April 21, 1982, through March 31, 1987[a].

| | | $\rho_k(R^o_{S,t}, R^o_{S,t-k})$ | | $\rho_k(R^o_{F,t}, R^o_{F,t-k})$ | | | $\rho_k(\epsilon_{S,t}, \epsilon_{S,t-k})$ | |
|---|---|---|---|---|---|---|---|---|
| Lag $k$ | No. of obs.[b] | $\hat{\rho}^c_k$ | $t(\hat{\rho}_k)^d$ | $\hat{\rho}^c_k$ | $t(\hat{\rho}_k)^d$ | No. of obs.[e] | $\hat{\rho}^c_k$ | $t(\hat{\rho}_k)^d$ |
| 1 | 86,952 | 0.5117 | 175.61 | 0.0229 | 6.77 | 84,454 | 0.0071 | 2.06 |
| 2 | 85,703 | 0.2654 | 80.60 | 0.0265 | 7.76 | 83,205 | 0.0053 | 1.52 |
| 3 | 84,454 | 0.1312 | 38.46 | 0.0015 | 0.45 | 81,956 | 0.0068 | 1.95 |
| 4 | 83,205 | 0.0759 | 21.96 | −0.0137 | −3.96 | 80,707 | 0.0050 | 1.41 |
| 5 | 81,956 | 0.0460 | 13.17 | −0.0222 | −6.36 | 79,458 | 0.0052 | 1.48 |
| 6 | 80,707 | 0.0199 | 5.64 | −0.0108 | −3.06 | 78,209 | −0.0042 | −1.18 |
| 7 | 79,458 | 0.0077 | 2.18 | −0.0087 | −2.46 | 76,960 | −0.0119 | −3.30 |
| 8 | 78,209 | 0.0154 | 4.32 | −0.0015 | −0.42 | 75,711 | 0.0017 | 0.46 |
| 9 | 76,960 | 0.0195 | 5.42 | 0.0039 | 1.07 | 74,462 | −0.0005 | −0.15 |
| 10 | 75,711 | 0.0110 | 3.04 | −0.0030 | −0.83 | 73,213 | −0.0082 | −2.22 |
| 11 | 74,462 | 0.0018 | 0.49 | 0.0047 | 1.29 | 71,964 | −0.0163 | −4.37 |
| 12 | 73,213 | 0.0019 | 0.51 | 0.0002 | 0.07 | 70,715 | −0.0067 | −1.77 |

*a*. The numbers in this table are taken from Stoll and Whaley (1990, Tables 1 and 3).

*b*. The number of observations used in the computation of the serial correlation coefficient. Note that as the lag $k$ is incremented by one, the number of observations lost equals the number of days in the sample period. This reflects the loss of one return each day of the sample. The serial correlation coefficient estimates are, therefore, not contaminated by using returns from adjacent days.

*c*. The estimated lag $k$ serial correlation coefficient across all five-minute returns in all days of the period, excluding overnight returns and the first two returns each trading day.

*d*. The $t$-ratio corresponding to the null hypothesis that $\rho_k$ equals zero.

*e*. The number of observations drops by 2,498 as a result of fitting an $ARMA(2,3)$ regression model to observed returns.

ficient of the lag one futures return, $\hat{\beta}_1$, is 0.1015, showing that there is a tendency for the futures market to lead the stock market. All other coefficients in the return innovation regression are indistinguishably different from zero in an economic sense. When stock index returns are used as the dependent variable, the leading effect of the futures market appears considerably longer, but most of this is illusion attributable to infrequent trading in the stock market. Overall, the evidence supports the notion that futures markets tend to play a price discovery role in the marketplace.

TABLE 7.7 Parameter estimates from regressions of S&P 500 index returns/return innovations on lag, contemporaneous, and lead nearby S&P 500 futures returns for the 1249-day period April 21, 1982 through March 31, 1987[a].

$$\text{Returns:} \quad R_{S,t} = \alpha + \sum_{k=-3}^{3} \beta_k R_{F,t-k} + u_t$$

$$\text{Return innovations:} \quad \epsilon_{S,t} = \alpha + \sum_{k=-3}^{3} \beta_k R_{F,t-k} + u_t$$

| | Returns | | Returns Innovations | |
|---|---|---|---|---|
| No. of Obs. | 78,209 | | | 78,209 |
| $R^2$ | 0.4730 | | | 0.2132 |
| | Parameter estimate[b] | $t$-ratio[c] | Parameter estimate[b] | $t$-ratio[c] |
| $\hat{\alpha}$ | -0.0001 | -1.08 | -0.0002 | -1.73 |
| $\hat{\beta}_{-3}$ | -0.0077 | -6.57 | -0.0094 | -8.04 |
| $\hat{\beta}_{-2}$ | -0.0158 | -13.48 | -0.0153 | -13.04 |
| $\hat{\beta}_{-1}$ | 0.0213 | 18.10 | 0.0194 | 16.54 |
| $\hat{\beta}_{0}$ | 0.1690 | 142.93 | 0.1338 | 113.50 |
| $\hat{\beta}_{1}$ | 0.2032 | 171.14 | 0.1015 | 85.72 |
| $\hat{\beta}_{2}$ | 0.1330 | 111.45 | 0.0153 | 12.87 |
| $\hat{\beta}_{3}$ | 0.0798 | 66.50 | 0.0059 | 4.92 |

*a*. The numbers in this table are taken from Stoll and Whaley (1990b, Table 5).

*b*. Parameter estimates obtained from times series regression across all five-minute returns in all days of the period, excluding overnight returns and the first two returns each trading day.

*c*. The $t$-ratio corresponding to the null hypothesis that the respective coefficient equals zero.

## 7.5 HEDGING MARKET RISK

Stock index futures contracts are useful in a variety of risk management situations. In this section, we examine an important one—hedging market risk. Assume you are responsible for managing a $50,000,000 stock portfolio. This portfolio has a systematic risk coefficient ($\beta_p$) of 1.20 relative to the S&P 500 index and a total risk ($\sigma_p$) of forty percent on an annualized basis. The future value of the promised dividends on this stock portfolio over the next three months is $400,000, or 0.8 percent of the current portfolio value. At the same time, the S&P 500 stock index portfolio has a total risk level ($\sigma_S$) of twenty-five percent annually and promises cash dividends over the next three months amounting to one percent of the current index value. The current S&P 500 index value is 373.63 and the price of the nearby, three-month S&P 500 futures contract is 375.50. A three-month T-bill promises a 1.5 percent rate of return. This illustration assumes the cost of carry relation (7.11) holds, that is, $375.50 = 373.63(1 + .015 - .01)$.

Suppose that your research director has informed you that the market (as reflected by the S&P 500) will drop by sixteen percent over the next three months. You have a great deal of confidence in his prediction so you decide to hedge the market risk of your portfolio. One option that you have is to liquidate the stock portfolio and buy T-bills, however this strategy would not allow you to capture the non-market returns that your portfolio of "winners" is expected to earn over the next three months. Selling S&P 500 futures contracts, on the other hand, allows you to hedge the market risk of the stock portfolio without selling your stocks.

## Forming the Hedge Portfolio

The optimal number of futures contracts to sell in this instance can be obtained indirectly using the stock portfolio beta. $\beta_p = 1.20$ implies that the stock portfolio is expected to earn 1.2 times the gain/loss of the S&P 500 index per dollar invested. The stock portfolio beta is defined as

$$\beta_p \equiv \frac{\text{Cov}(\tilde{R}_p, \tilde{R}_S)}{\text{Var}(\tilde{R}_S)},$$

where $\tilde{R}_p$ and $\tilde{R}_S$ are the random rates of return on the stock portfolio and the market index (in this case, the S&P 500), respectively. To understand the relation between the stock portfolio beta and the optimal hedge ratio, we need to establish the relation between the futures and stock index returns over the hedge period, which is equal to the futures contract life in this illustration. Over the hedge period, the stock index return is

$$\tilde{R}_S = \frac{\tilde{S}_T - S_0}{S_0},$$

and the futures return is

$$\tilde{R}_F = \frac{\tilde{F}_T - F_0}{F_0}.$$

Using (7.11) to substitute for $F_0$,

$$\begin{aligned}\tilde{R}_F &= \frac{\tilde{S}_T - S_0(1+r^*-d^*)}{S_0(1+r^*-d^*)} \\ &= \frac{\tilde{R}_S}{1+r^*-d^*} - \frac{r^*-d^*}{1+r^*-d^*}.\end{aligned}$$

Rearranging to isolate $\tilde{R}_S$, we get

$$\tilde{R}_S = \tilde{R}_F(1+r^*-d^*) + r^* - d^*.$$

Substituting for the stock index return, the expression for the stock portfolio beta becomes

$$\begin{aligned}\beta_p &= \frac{\text{Cov}[\tilde{R}_p, \tilde{R}_F(1+r^*-d^*)+r^*-d^*]}{\text{Var}[\tilde{R}_F(1+r^*-d^*)+r^*-d^*]} \\ &= \frac{\text{Cov}(\tilde{R}_p, \tilde{R}_F)}{\text{Var}(\tilde{R}_F)(1+r^*-d^*)}.\end{aligned}$$

The remaining step in showing the relation between the stock portfolio beta and the hedge ratio involves substituting the relations between returns and price changes. These relations are $\tilde{R}_p \equiv \tilde{\Delta}_p/p_0$ and $\tilde{R}_F \equiv \tilde{\Delta}_F/F_0$. Hence, the stock's rate of return beta $\beta_p$ is

$$\begin{aligned}\beta_p &= \frac{\text{Cov}(\tilde{\Delta}_p/p_0, \tilde{\Delta}_F/F_0)}{\text{Var}(\tilde{\Delta}_F/F_0)(1+r^*-d^*)} \\ &= \frac{\frac{1}{p_0 F_0}\text{Cov}(\tilde{\Delta}_p, \tilde{\Delta}_F)}{\frac{1}{F_0^2}\text{Var}(\tilde{\Delta}_F)(1+r^*-d^*)} \\ &= \frac{\text{Cov}(\tilde{\Delta}_p, \tilde{\Delta}_F)F_0}{\text{Var}(\tilde{\Delta}_F)(1+r^*-d^*)p_0}.\end{aligned}$$

Using the definition of the optimal hedge ratio given in Chapter 4 and assuming the cost-of-carry relation, (4.9), $\beta_p$ can be written as

$$\beta_p = -h^* \frac{S_0}{p_0}.$$

Finally, the initial investment in the stock portfolio and the cash index with respect to which $\beta_p$ is calculated are the same, so $p_0 = S_0$. This implies that

$$\beta_p = -h^*.$$

In other words, the optimal hedge ratio is the negative of the stock portfolio beta. In the case of the example, the optimal hedge ratio is

$$h^* = -1.2.$$

The optimal number of futures contracts to sell is therefore the stock portfolio beta times the number of units of the stock portfolio,

$$1.2\left[\frac{50,000,000}{373.63(500)}\right] = 321.17.$$

## Assessing Hedging Effectiveness

The information indicates that the variance of the unhedged stock portfolio return is $.40^2 = .16$. If the stock portfolio investment is one dollar, the *price change* variance of the unhedged portfolio is also .16. In Chapter 4, we learned that hedging effectiveness is measured by the adjusted R-squared of the regression of cash price changes on futures price changes. The R-squared, in turn, is closely related to the correlation of cash price changes with futures price changes (i.e., $R^2 = \rho^2_{p,F}$). To find the effectiveness of the S&P 500 hedge in our illustration, therefore, we focus on the correlation coefficient between the stock portfolio price changes and the futures price changes. Over the life of the futures contract, the futures and stock index price changes are perfectly correlated and the standard deviation of the futures price change equals the standard deviation of the stock index price change, so the correlation coefficient may be written

$$\rho_{p,F} = \rho_{p,S} = \frac{\sigma_{p,S}}{\sigma_p \sigma_S}.$$

Also, we know that $\beta_p = \sigma_{p,S}/\sigma^2_S$, so

$$\rho_{p,F} = \beta_p \frac{\sigma_S}{\sigma_p}.$$

On the basis of the given values ($\beta_p = 1.20$, $\sigma_S = 0.25$ and $\sigma_p = 0.40$); the correlation coefficient, $\rho_{p,F}$, is 0.75. The R-squared is thus 0.5625, and the proportion of the stock portfolio return variance that is unrelated to the return variance of index futures is $1 - .5625$ or 0.4375. The remaining variance of the rate of price change on the hedged portfolio is therefore

$$\sigma^2_h = 0.4375(.16) = .07.$$

## Decomposing the Hedge Portfolio Return

Suppose that the S&P 500 index drops by twenty percent over the three-month period after the hedge portfolio is formed. Over the same time, your stock portfolio drops to a value of $40,000,000, excluding dividends. Find the overall rate of return on your hedged portfolio, and decompose the overall return into its riskless rate

TABLE 7.8 Hedging market risk of a stock portfolio that has a $\beta = 1.2$ and a three-month dividend yield of 0.8 percent.

| | Cash Market | | December Futures | | |
|---|---|---|---|---|---|
| | Index Level | Value of Stock Portfolio | Futures Price | Value of Futures Position[a] | Value of Hedged Portfolio |
| Sept 15 | 373.63 | 50,000,000 | 375.50 | –60,300,000 | |
| Dec 15 | 298.90 | 40,400,000[b] | 298.90 | –48,000,000 | |
| Gain | | –9,600,000 | | 12,300,000 | 2,700,000 |
| Return(%) | –20.00 | –19.20[c] | –20.40 | –24.60[c] | 5.40[c] |

*a.* The optimal hedge involves selling 321.17 futures contracts, with each contract valued at 500 times the index futures price.

*b.* Includes dividends of $400,000.

*c.* Dollar gain divided by the initial stock portfolio value, $50,000,000.

and abnormal return components. Table 7.8 provides such a decomposition for a hedge established on September 15 and liquidated on December 15, when the futures contract is assumed to expire.

The overall rate of return on the unhedged portfolio can be measured easily by focusing on the price appreciation and dividend yield components of total return, that is,

$$R_p = \left(\frac{40,000,000}{50,000,000} - 1\right) + \left(\frac{400,000}{50,000,000}\right) = -19.20\%.$$

To find the hedged portfolio return, we must also compute the rate of return on the futures. At the outset, the S&P 500 index level was 373.63 and the three-month S&P 500 futures price was 375.50. If the S&P 500 index level fell by twenty percent over the three-month period, the new index level and futures price (recall that futures had three months to expiration when they were sold) are 373.63(.80) or 298.90. The rate of return on the index futures over the period was therefore

$$R_F = \frac{298.90}{375.50} - 1 = -20.40\%.$$

Thus, the total return of the hedged portfolio over the three-month period is

$$\begin{aligned} R_h &= R_p + h\left(\frac{F_0}{S_0}\right)R_F \\ &= -0.1920 - 1.2\left(\frac{375.50}{373.63}\right)(-0.2040) \\ &= 5.40\%. \end{aligned}$$

The alternative to hedging in this example is to liquidate the stock portfolio and buy three-month T-bills. Such an action would have produced a 1.5 percent return, given our assumption that the T-bill rate is 1.5 percent. The riskless rate and abnormal performance components of the hedge portfolio return in this illustration are therefore 1.5 percent and 3.9 percent, respectively. In other words, the 3.9 percent return was the abnormal or extra-market rate of return arising from the fact that the portfolio of "winners" outperformed the market on a risk-adjusted basis.

The hedged portfolio would also have earned 1.5 percent if the stock portfolio had declined exactly according to its beta of 1.20, without an abnormal return. In that case, the return would have been

$$R_p = r^* + (R_m - r^*)\beta_p = 0.015 + (-0.19 - 0.015)1.2 = -0.231,$$

where $R_m$ is the return on the stock index including the dividends, or $-0.19$. That implies a value for the stock portfolio, including dividends, of $38,450,000, instead of the value of $40,400,000 shown in Table 7.8. The values in Table 7.8 for the cash index and the futures market would remain the same. The dollar gain on the hedged portfolio becomes $750,000, and the hedged return becomes 1.5 percent, exactly the same as the riskless rate.

It is worth noting that the hedged stock portfolio has basis risk because the portfolio's return is not perfectly correlated with the index futures return. If, for example, the stock portfolio had a negative abnormal return, the hedged portfolio would have earned less than the riskless rate.

## 7.6 SUMMARY

In this chapter, stock index futures contracts and the composition of stock indexes underlying futures contracts are described. The cost-of-carry relation for stock indexes is derived, and the role of index arbitrage in maintaining the link between stock index futures and cash prices is explained. Evidence on the short-run behavior of the returns of index futures and of the cash index is presented. Finally, the use of stock index futures to hedge the market risk in a stock or a portfolio of stocks is illustrated in detail.