INNOVATIONS

Valuing S&P 500 Bear Market Warrants with a Periodic Reset

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S&P 500 index bear market warrants with a three-month reset are new structured products on the Chicago Board Options Exchange and the New York Stock Exchange. They are like index puts with one difference: The exercise price resets automatically at a higher level if the index level is above the original exercise price on the reset date three months later.

This study develops valuation methods for bear market warrants with a periodic reset, examines the warrant's risk characteristics, and compares observed warrant prices with those of comparable S&P 500 puts.

Late in 1996, the Chicago Board Options Exchange (CBOE) and the New York Stock Exchange (NYSE) began trading S&P 500 index bear market warrants with a three-month reset. These new structured products are one-shot, privately issued securities offered for sale by the International Finance Corporation (IFC). Both the CBOE and the NYSE agreed to make markets in these securities, which are like index puts except that the exercise price is automatically reset at a higher level if the index level is above the original exercise price on the reset date three months after the original issuance.

Among the potential users of this new product are portfolio insurers, who buy new puts to protect their stock portfolio gains each time the stock market advances significantly. S&P 500 bear market warrants ratchet up the face value of the portfolio insurance automatically as the market rises.

The purpose of this article is to develop valuation methods for bear market warrants with a periodic reset, examine the warrant's risk characteristics, and compare observed warrant prices with those of comparable S&P 500 puts.

I. DESCRIPTION AND VALUATION

The contract specifications of the IFC's bear market warrant entitle its holder, upon exercise, to \$50 times the relative amount that the warrant is in the money (i.e., the ratio of the exercise price less the index level to the exercise price). The original exercise price of the warrant is set to be the same as the closing index level on the date of the warrant's issuance. The exercise price is reset at the closing index level on the reset date (three months after issue) if that index level is above the original exercise price.

To understand how to value a bear market warrant, first consider its terminal value at expiration, W_T, that is:

$$W_T = \begin{cases} \dfrac{S_t - S_T}{S_t} & \text{if } S_t > X,\ S_T \le S_t \\[2ex] \dfrac{X - S_T}{X} & \text{if } S_t \le X,\ S_T \le X \\[2ex] 0 & \text{if } (S_t > X \text{ and } S_T > S_t) \\ & \quad \text{or } (S_t \le X \text{ and } S_T > X) \end{cases} \qquad (1)$$

where S_t is the closing index level on the reset date, S_T is the closing index level on the expiration date, and X is the original exercise price set on the date of issuance. The time subscripts t and T are the reset date and the expiration date of the warrant.

This is a general expression for the payoff of a bear market reset warrant, where, for expositional convenience, we are temporarily ignoring the \$50 contract multiplier. In the first case in (1), the warrant is in the money at expiration ($S_T \le S_t$), after having its original exercise price reset at the index level at time t because the index level exceeded the original exercise price ($S_t > X$). In the second case in (1), the warrant is in the money at expiration ($S_T \le X$), but the exercise price is not reset because the index level is below the original exercise price on the reset date ($S_t \le X$). In the final case in (1), the warrant is out of the money at expiration, independent of whether the warrant's exercise price is reset.

Assuming that a risk-neutral hedge may be formed between the warrant and the underlying index, the bear market warrant can be valued using risk-neutral valuation. Under risk neutrality, the current value of the bear market warrant with a periodic reset equals the present value, at the risk-free rate, of the expected terminal value of the warrant. The expected terminal value of the warrant, in turn, is the sum of the expected conditional terminal values as defined by (1).

Valuing the bear market warrant, therefore, involves evaluating the terms of the right hand-side of (2), that is:

$$\begin{aligned} W &= e^{-rT} E\left(\frac{S_t - S_T}{S_t} \mid S_t > X,\ S_T \le S_t \right) \times \\ &\quad \Pr(S_t > X,\ S_T \le S_t) + \\ &\quad e^{-rT} E\left(\frac{X - S_T}{X} \mid S_t \le X,\ S_T \le X \right) \times \\ &\quad \Pr(S_t \le X,\ S_T \le X) \\ &= W1 + W2 \end{aligned} \qquad (2)$$

where W1 is the present value of the expected terminal value of the warrant conditional on the warrant's exercise price being reset and the warrant being in the money at expiration times the probability that the warrant's exercise price is reset and the warrant is in the money at expiration; and W2 is the present value of the expected terminal value of the warrant conditional on the warrant's exercise price *not* being reset and the warrant being in the money at expiration times the probability that the warrant's exercise price is *not* reset at time t and the warrant is in the money at expiration. In all cases, expectations and probabilities are under the risk-neutral measure.

Under the Black-Scholes [1973] framework, the price of the asset underlying the option is assumed to follow geometric Brownian motion:

$$dS/S = \mu dt + \sigma dz$$

where μ is the expected return on the index, σ is the standard deviation of the index return, and z is a Wiener process.

Among other things, this implies that index returns are serially independent and that the asset price is lognormally distributed at any future time τ. Also, in a risk-neutral environment, the expected return on the asset equals the risk-free interest rate less the asset's dividend yield, that is, $\mu = r - d$. We apply these assumptions and results to value the bear market warrant by evaluating W1 and W2 in (2).

Focusing first on W1, we separate the terms in the difference and simplify as follows:

$$W1 = e^{-rT}\Pr(S_t > X, S_T \leq S_t) - e^{-rT} \times$$
$$E[(S_T/S_t) \mid S_t > X, S_T \leq S_t] \times$$
$$\Pr(S_t > X, S_T \leq S_t) \tag{3}$$

The second term in (3) can be further simplified. Defining $x = S_T/S_t$, the conditional expectation may be rewritten

$$E(x \mid S_t > X, x \leq 1) = E(x \mid x \leq 1) \tag{4A}$$

This follows because the probability that the stock price will double over the next nine months (for example) is independent of the current level of the stock price. That is, knowing that the stock price is above X at time t tells us nothing about the return on the stock from t to T. The independence of returns also implies that the probability expression in the second term on the right-hand side can be written as

$$\Pr(S_t > X, x \leq 1) = \Pr(S_t > X)\Pr(x \leq 1) \tag{4B}$$

This equation describes the probability that the stock price will be above X at time t *and* that it will subsequently fall from time t to time T. Since we have established that these two events are independent, the probability of the joint event is simply the product of the two individual probabilities.

The lognormality assumption permits further simplification. Since S_t and S_T are both lognormally distributed, x (their ratio) is also lognormal, and the natural logarithm of x, ln(x), is normal with mean $(r - d - 0.5\sigma^2)(T - t)$, and variance $\sigma^2(T - t)$. This follows directly from the properties of a lognormal distribution.

Consequently, the value of W1 may be written

$$W1 = e^{-rT}\Pr(S_t > X)\Pr(x \leq 1) -$$
$$e^{-rT}E(x \mid x \leq 1)\Pr(S_t > X)\Pr(x \leq 1)$$
$$= e^{-rT}N_1(a_2)N_1(-c_2) -$$
$$e^{-d(T-t)}N_1(a_2)N_1(-c_1)e^{-rt} \tag{5}$$

where

$$a_1 = \frac{\ln(S/X) + (r - d + 0.5\sigma^2)t}{\sigma\sqrt{t}}.$$
$$a_2 = a_1 - \sigma\sqrt{t}.$$
$$c_1 = \frac{(r - d + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$
$$c_2 = c_2 = c_1 - \sigma\sqrt{T - t}.$$

where $N_1(a)$ is a cumulative univariate normal distribution function with upper integral limit a.

Valuing W2 in (2) is not quite as straightforward as the first term, in the sense that, although asset returns are independent, asset prices are not. Consequently, the conditional expectation and probability expressions do not simplify into univariate normal expressions. Nonetheless, under the assumptions of the Black-Scholes model, the expressions in W2 are tractable and may be written

$$W2 = e^{-rT}E[(X/X) \mid S_t \leq X, S_T \leq X]\Pr(S_t \leq X, S_T \leq X)$$
$$- e^{-rT}E[(S_t/X) \mid S_t \leq X, S_T \leq X]\Pr(S_t \leq X, S_T \leq X)$$
$$= e^{-rT}\Pr(S_t \leq X, S_T \leq X)$$
$$- e^{-rT}(1/X)E[S_T \mid S_t \leq X, S_T \leq X] \times$$
$$\Pr(S_t \leq X, S_T \leq X)$$
$$= e^{-rT}N_2(-a_2, -b_2, \sqrt{t/T}) -$$
$$(S/X)e^{-dT}N_2(-a_1, -b_1, \sqrt{t/T}) \tag{6}$$

where $N_2(a, b, \rho)$ is a cumulative bivariate normal distribution function with upper integral limits a and b and correlation coefficient ρ,

$$b_1 = \frac{\ln(S/X) + (r - d + 0.5\sigma^2)T}{\sigma\sqrt{T}}.$$

$$b_2 = b_1 - \sigma\sqrt{T}.$$

Equation (6) makes use of the fact that 1) the exercise price is known and can be treated as a constant, and 2) S_t and S_T are jointly lognormal so their natural logarithms are jointly normal. This yields an expression in terms of the bivariate normal distribution. Combining Equations (5) and (6), the value of a European-style bear market warrant with a periodic reset is

$$W = N_1(a_2)N_1(-c_2) - e^{-d(T-t)}N_1(a_2)N_1(-c_1)e^{-rt} + e^{-rT}N_2(-a_2, -b_2, \sqrt{t/T}) - (S/X)e^{-dT}N_2(-a_1, -b_1, \sqrt{t/T}) \qquad (7)$$

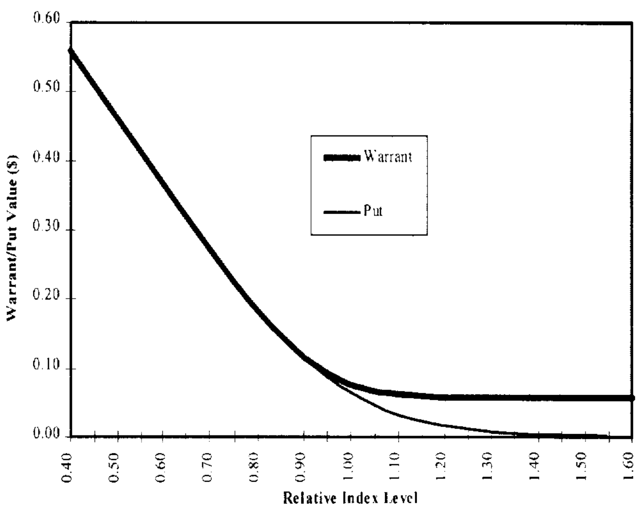
II. RISK CHARACTERISTICS

Perhaps the best way to understand the bear market warrant vis-à-vis an index put is to compare their values as the valuation parameters change. To do so, assume that both the warrant and the put have 365 days to expiration. Initially, the index level and the exercise price of both instruments are 1. The warrant's exercise price may be reset in 90 days should the index level exceed 1. The interest rate is assumed to be 5% annually; the dividend yield on the index portfolio is 2%; and the volatility rate of the index portfolio is 20%.

Exhibit 1 illustrates the difference between the two instruments. As the index level falls below 1, the value of the warrant approaches the value of the put. The reason is, of course, that the chances of the warrant's exercise price being reset are diminished. On the other hand, as the index level increases, the value of the warrant falls but not by as much as the put.

Note that as the relative index level increases, the value of the warrant asymptotes toward 0.056.

EXHIBIT 1
WARRANT AND PUT VALUES



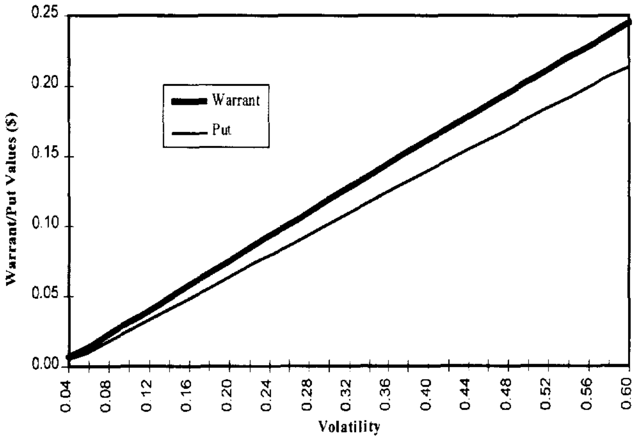
This value is the present value of a nine-month at-the-money put, which is what the warrant becomes when reset. As the relative index level rises, reset becomes more and more likely to occur.

Exhibit 2 illustrates the effect that changes in volatility have on the warrants and a standard index put. The model inputs are the same as in Exhibit 1, and at-the-money warrant and put values are computed for a range of index volatilities. Exhibit 2 clearly demonstrates that the value of the warrant is more sensitive to changes in volatility. As volatility increases, both the warrant and the put become more valuable, as there is more chance of the option finishing well in the money. In addition, the warrant becomes even more valuable, since higher volatility also means higher probability of the exercise price being reset, in which case the payoff at maturity will be greater than under a standard put.

Exhibit 3 shows the relationship between the warrant and put deltas and the relative index level. Put option deltas generally indicate the number of shares that must be purchased in order to hedge an option position. In this case, however, the warrant is based on the ratio of the current index level to the index level when the warrant was initiated. Therefore, the deltas illustrated in Exhibit 3 are in terms of this ratio.

To convert this to a hedge ratio in terms of units

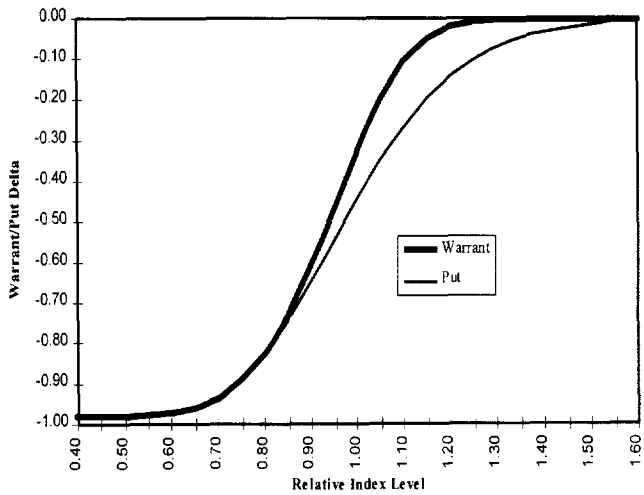
EXHIBIT 2
WARRANT AND PUT VALUES AND VOLATILITY



of the index, the hedge ratios in Exhibit 3 must simply be multiplied by the initial index level. Note that the warrant delta is zero for any relative index level beyond about 1.20. In this range, reset is inevitable, in which case the warrant is really just a claim to receive an at-the-money put on the reset date, regardless of any small changes in the index prior to reset.

For a range of volatilities and index values relative to the original exercise price, Exhibit 4 quantifies

EXHIBIT 3
WARRANT AND PUT DELTAS



the "reset premium," which we define as the premium that must be paid for a reset warrant (over and above the price of an otherwise identical, but not resettable instrument) expressed as a percentage of the price of the non-resettable instrument. Two features are apparent from the table.

First, as the stock price increases above the original exercise price (S/X increases), the reset warrant becomes much more valuable than a non-resettable instrument. This is because it becomes more likely that the reset feature will be activated so that the reset warrant will become an at-the-money instrument on the reset date. Conversely, the non-resettable instrument becomes likely to expire out of the money and consequently has a very low value.

Second, this effect is mitigated when volatility is high. Higher volatility means a greater range of possible movements in the price of the underlying. Hence, when volatility is high, the non-resettable instrument has a greater chance of finishing in the money and is more valuable than when volatility is low.

III. AMERICAN-STYLE BEAR MARKET WARRANTS

The valuation equation and our discussion presume that the bear market warrant is a European-style instrument. The European-style feature permits an analytical valuation formula to be derived. With American-style warrants such as IFC's S&P 500 bear market warrant, analytical solutions are generally not possible, and numerical methods must be used. The most popular American-style option valuation approximation method is the Cox, Ross, and Rubinstein [1979] binomial lattice procedure. This procedure may be adapted to value American-style bear market warrants.

Applying the binomial lattice procedure to value American-style options is described in most standard options textbooks (e.g., Stoll and Whaley [1993, pp. 202-206]; Hull [1996, pp. 201-204]). The modifications necessary to value bear market warrants are as follows.

First, the binomial lattice is set up so that the warrant's life is assumed to end at the reset date.

Second, in place of using the option's terminal boundary condition to value the option at each asset price node, we use another binomial lattice. For each

EXHIBIT 4
PERCENTAGE RESET PREMIUM FOR BEAR MARKET WARRANTS

	Volatility										
S/X	0.150	0.175	0.200	0.225	0.250	0.275	0.300	0.325	0.350	0.375	0.400
0.80	0.01	0.03	0.09	0.18	0.31	0.48	0.68	0.90	1.13	1.62	1.62
0.85	0.12	0.28	0.52	0.82	1.16	1.52	1.88	2.24	2.60	2.93	3.25
0.90	1.02	1.61	2.23	2.84	3.42	3.96	4.45	4.89	5.29	5.64	5.96
0.95	5.49	6.47	7.28	7.93	8.47	8.90	9.26	9.55	9.79	9.98	10.13
1.00	20.57	19.79	19.16	18.63	18.17	17.76	17.39	17.05	16.73	16.43	16.15
1.05	58.97	49.26	42.83	38.26	34.84	32.18	30.05	28.28	26.80	25.53	24.43
1.10	141.03	105.93	84.83	70.96	61.22	54.04	48.55	44.22	40.71	37.81	35.37
1.15	302.06	205.46	154.12	121.97	100.58	85.50	74.38	65.87	59.19	53.80	49.38
1.20	610.72	378.66	264.04	198.49	157.15	129.18	109.23	94.42	83.05	74.09	66.87

of the asset price nodes in excess of the original exercise price, the binomial lattice procedure is used to value an at-the-money American-style put. The put is at the money since the warrant's exercise price is reset at the prevailing asset price. Put option valuation is appropriate at that time, because after the reset date the warrant is identical to a standard put. For each of the asset price nodes below the original exercise price, the binomial lattice is used to value an American-style put with an exercise price of X.

Finally, with the warrant values at all the nodes on the reset date identified, the valuation procedure proceeds backward through time, one step at a time, with each new node computed as the present value of the expected value of the two nodes directly in front. Each of these newly computed warrant values is checked for the prospect of early exercise, and, if early exercise is optimal, the exercise proceeds replace the computed value at that node. The iterative procedure proceeds backward until the current value of the warrant is identified. This procedure is illustrated in Exhibit 5.

IV. APPLYING THE MODEL TO IFC BEAR MARKET WARRANTS

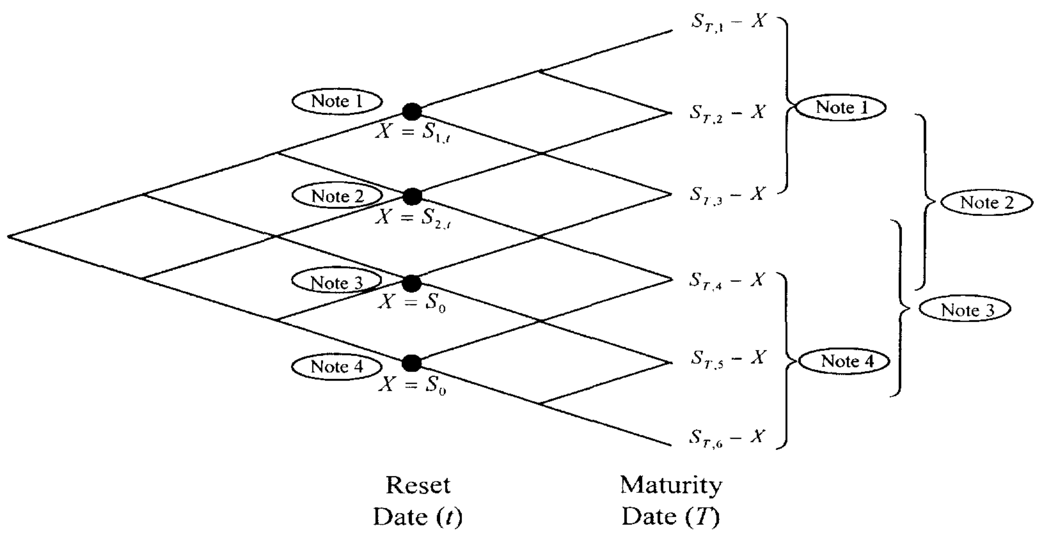
The International Finance Corporation issued S&P 500 bear market warrants with a three-month reset on November 20, 1996. The exercise price was initially set equal to 743.5, the closing index level on November 20, 1996. According to the terms of the warrant, the reset date is February 20, 1997, and the expiration date is November 20, 1997. Upon exercise, the warrant holder receives $50 times the ratio of the exercise price less the index level to the exercise price. The warrants are American-style.

To analyze the market pricing of the ITC's bear market warrants, we examine all trades on the CBOE and NYSE from the first day of trading, November 21, 1996, through March 31, 1997. The data were provided by the CBOE. Each time-stamped trade record gives the trade price and the contract volume. The total number of contracts traded during the four-month interval was 14,565,900.

Rather than examining warrant pricing directly, we use the binomial warrant valuation methodology outlined above to compute implied volatilities. A number of model inputs are necessary. For the index level, we use the last reported S&P 500 index level prior to the time of trade. For the interest rate, we use the rate implied by the bid and ask discounts of the November 1997 T-bill. The bid and ask discounts come from various issues of the *Wall Street Journal*, and the midpoints of the bid and ask discounts are used to compute the continuously compounded interest rates. The continuously compounded dividend yield rates for the S&P 500 stock index portfolio are computed from the simple daily dividend yields reported in various issues of the "Standard & Poor's 500 Information Bulletin." During the sample period, the warrant-implied volatilities averaged 21.7%, and ranged from 18.3% to 26.3%.

To benchmark the level of the warrant-implied volatilities, implied volatilities are computed using

EXHIBIT 5
BINOMIAL VALUATION PROCEDURE FOR AMERICAN-STYLE BEAR MARKET RESET WARRANTS



Note 1: On the reset date, the stock price has risen, so the exercise price is reset to the current stock price $S_{1,t}$ where the notation signifies that this is the stock price at the top node on date t. The valuation of the option at this point proceeds in the standard way. From the top node at date t, three nodes can be reached at date T (nodes 1, 2, and 3). The payoff at each node will be the stock price at that node less the appropriate exercise price.

Note 2: On the reset date, the stock price has risen, so the exercise price is reset to the current stock price $S_{2,t}$ where the notation signifies that this is the stock price at the second top node on date t. The valuation of the option at this point proceeds in the standard way. From the second top node at date t, three nodes can be reached at date T (nodes 2, 3, and 4). The payoff at each node will be the stock price at that node less the appropriate exercise price. Note that the exercise price used for nodes 2 and 3 will be different from that employed under Note 1.

Note 3: On the reset date, the stock price has fallen, so the exercise price is not reset and remains equal to the stock price at date 0, when the contract was entered into. The valuation of the option at this point proceeds in the standard way. From the second bottom node at date t, three nodes can be reached at date T (nodes 3, 4, and 5). The payoff at each node will be the stock price at that node less the appropriate exercise price. Note that the exercise price used for nodes 3 and 4 will be different from that employed under Note 2.

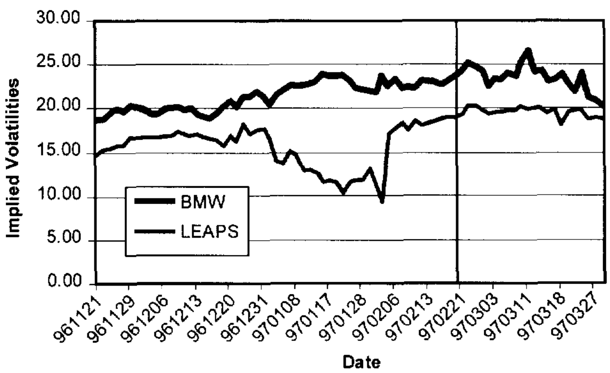
Note 4: On the reset date, the stock price has fallen, so the exercise price is not reset and remains equal to the stock price at date 0, when the contract was entered into, as in Note 3.

trade prices of the December 1997 S&P 500 put option leaps with exercise prices of 72.5, 75.0, and 77.5. Leaps rather than S&P 500 index options are a more appropriate benchmark because the S&P 500 leap's contract size (i.e., one-tenth the index level) and time to expiration more closely match those of the bear market warrant. At-the-money leaps are used to control for the fact that the warrants are at the money at the time of issuance.

To compare the two sets of implied volatilities, we compute the average implied volatility across all trades of the bear market reset warrants each day during the sample period. We do the same for the December 1997 S&P 500 leaps. The results are shown in Exhibit 6. The graph clearly shows that the bear market reset warrants trade at a premium to the leaps, and the premium seems to persist even after the reset date of February 20, 1997, when the exercise price was reset to 802.80.

To gauge the magnitude of the pricing premium, we use the average leap-implied volatility each day to value the warrants traded on that day. We then calculate the price premium as the ratio of the warrant's trade price less the theoretical value to the theo-

EXHIBIT 6
IMPLIED VOLATILITIES OF BEAR MARKET RESET WARRANTS (BMW) AND AT-THE-MONEY S&P 500 LEAPS



Note: The reset date, February 20, 1997, is marked with the vertical line.

retical value. The average price premium across the warrant trades in the sample is 40.6%. Reassuringly, the size of the price premium has become smaller through time. Indeed, the average price premium since the reset date is only 20.6%. The premium has not disappeared, however. One possible explanation is that it is difficult, if not impossible, to short-sell the bear market warrants.

V. SUMMARY AND CONCLUSIONS

We have derived a valuation equation for a European-style bear market warrant with a single reset date. We use the valuation equation to examine the risk characteristics of the warrant and compare them to comparable index puts. We also describe a binomial lattice procedure that can be used to value the American-style version of the bear market warrants. We use the valuation procedure to compute the implied volatilities from the trade prices of the IFC S&P 500 bear market warrants during their first four months of trading.

We find that the implied volatilities on the warrants are about 500 basis points higher than comparable S&P 500 index leaps. Using the implied volatilities of the leaps, we then value the warrants to find that they have about a 40% price premium.

The size of the warrant price premium is likely driven by a combination of factors. First, unlike standard options that can be created at will each time a buyer and a seller agree on a price, bear market warrants are issued in fixed quantities. Consequently, there may be a fixed-supply premium embedded in the warrant price.

Second, with a fixed supply of warrants and a relatively inactive secondary market, short-selling bear market warrants is difficult if not impossible. Without arbitrage, the market prices of the warrants and the leaps need not come into the alignment suggested by the (arbitrage-based) valuation methodology.

Third, basing the payoff on the ratio of the index level to the exercise price obfuscates the similarities of the warrant and a standard put option. A simpler contract design would make the bear market warrants more straightforward to value, more clearly highlighting the value of the reset premium and potentially generating greater market interest. In addition, increasing the contract size would likely increase institutional interest in the product, and adding more reset dates might make the contract more valuable from a portfolio insurance standpoint.

ENDNOTE

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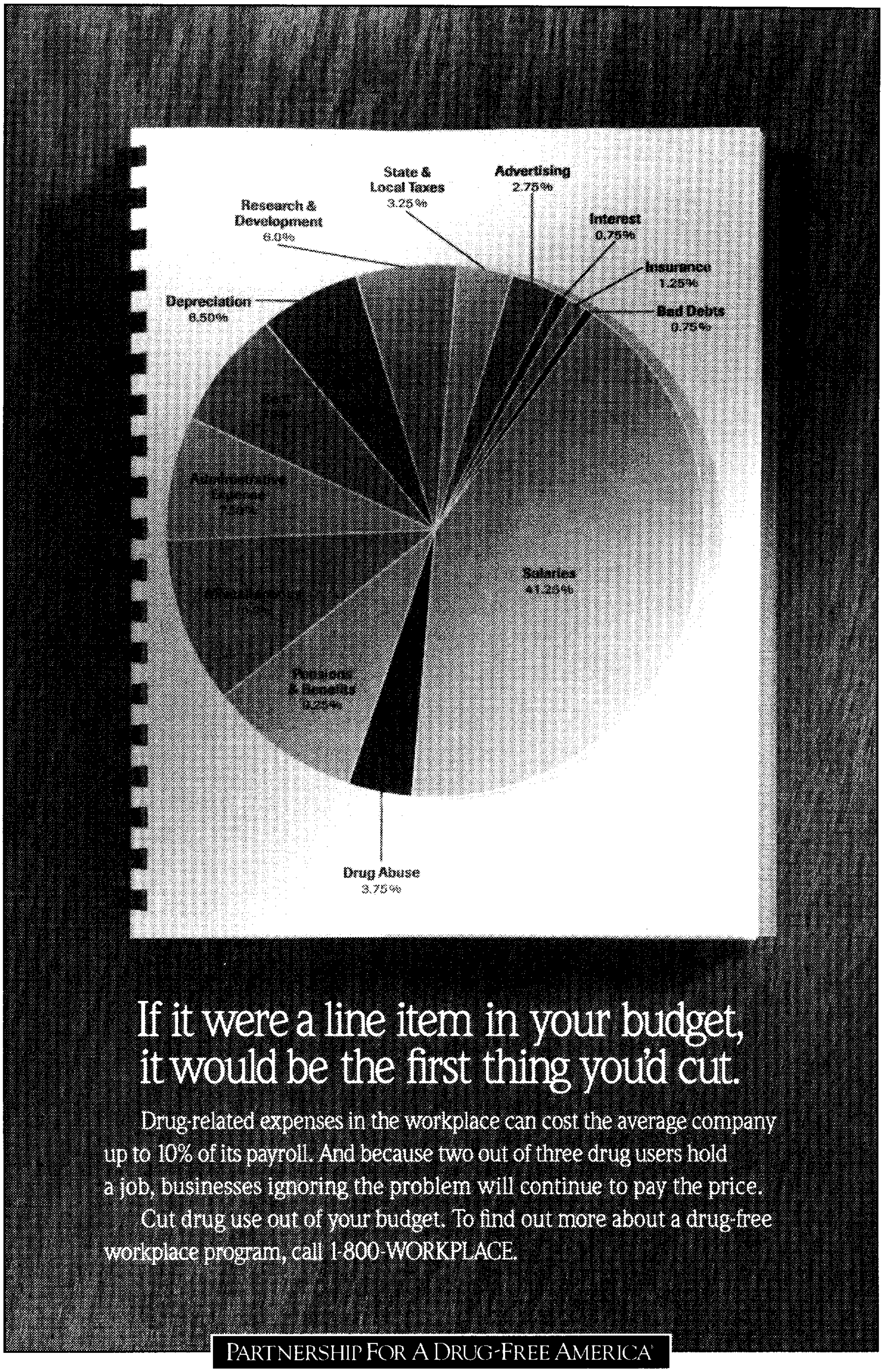
REFERENCES

Black, Fischer, and Myron Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81 (1973), pp. 637-659.

Cox, John C., Stephen Ross, and Mark Rubinstein. "Option Pricing: A Simplified Approach." *Journal of Financial Economics*, 7 (1979), pp. 229-263.

Hull, John C. *Options, Futures, and Other Derivative Securities*, 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1996.

Stoll, Hans R., and Robert E. Whaley. *Futures and Options: Theory and Applications*. Cincinnati: SouthWestern Publishing, 1993.


State &
Local Taxes
3.25%
Advertising
2.75%
Research &
Development
6.0%
Interest
0.75%
Insurance
1.25%
Depreciation
8.50%
Bad Debts
0.75%
Salaries
41.25%
Pensions
& Benefits
9.25%
Drug Abuse
3.75%
If it were a line item in your budget,
it would be the first thing you'd cut.
Drug-related expenses in the workplace can cost the average company up to 10% of its payroll. And because two out of three drug users hold a job, businesses ignoring the problem will continue to pay the price.
Cut drug use out of your budget. To find out more about a drug-free workplace program, call 1-800-WORKPLACE.
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