# DER 07.2

Stock index derivatives

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## Stock index derivatives

- Context:
  - Same derivatives valuation and risk management tools apply to all underlying reference assets.
    - Stock index derivatives are no exception.
- Purpose:
  - Provide history of stock index products.
  - Show stock index construction.
  - Review stock index arbitrage.
  - Cross-hedge stock market risk.

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## Stock index futures

- Early history of listings in US
  - Value Line futures (KCBT Feb 1982)
  - S&P 500 futures (CME April 1982)
  - Major Market Index futures (CBT July 1984)
  - DJIA futures (CBT Oct 1997)
- Other countries:
  - All Ordinaries (SFE Feb 1983)
  - TSE 300 (TSX Jan 1984)
  - FT-SE 100 (LIFFE May 1984)

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## Stock index futures

- Major derivatives exchanges often have futures contracts on multiple indexes.
  - "First-mover" usually gathers lion's share of trading volume.

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# Exchange products

| Product name | Index | Volume | Open interest | Turnover |
|---|---|---|---|---|
| E-mini S&P 500 Futures | S&P | 1,757,340 | 2,205,411 | 79.7% |
| E-mini Russell 2000 Index Futures | Russell | 227,943 | 553,319 | 41.2% |
| Adjusted Interest Rate S&P 500 Total Return Futures | S&P | 9,537 | 406,185 | 2.3% |
| E-mini Nasdaq-100 Futures | Nasdaq | 681,273 | 274,422 | 248.3% |
| S&P 500 Annual Dividend Index Futures | S&P | 5,775 | 258,634 | 2.2% |
| Micro E-mini S&P 500 Index Futures | S&P | 919,400 | 158,416 | 580.4% |
| Micro E-mini Nasdaq-100 Index Futures | Nasdaq | 1,094,657 | 106,242 | 1030.3% |
| Nikkei/Yen Futures | International Indices | 36,888 | 65,112 | 56.7% |
| S&P 500 Total Return Index Futures | S&P | 0 | 55,106 | 0.0% |
| S&P 500 Quarterly Dividend Index Futures | S&P | 1,338 | 46,630 | 2.9% |
| E-mini S&P MidCap 400 Futures | S&P | 11,422 | 39,272 | 29.1% |
| E-mini Financial Select Sector Futures | Select Sectors | 377 | 36,718 | 1.0% |
| E-mini Energy Select Sector Futures | Select Sectors | 759 | 32,697 | 2.3% |

Highest trading volume and open interest

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Variety of different domestic, international, and sector indexes

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Low values are buy-and-hold futures.
High values are day-trading futures.

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# Exchange products

- Support file: Index return statistics.xlsx
  - Downloaded daily data from Bloomberg for period 20121231 through 20221230.
    - Eliminated non-US trading days.

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# Stock index returns

| Summary statistics | | | | | | | |
|---|---|---|---|---|---|---|---|
| Description | SPX | RTY | NDX | INDU | MID | DJUSRE | S5FINL |
| *n* | 2,518 | 2,518 | 2,518 | 2,518 | 2,518 | 2,518 | 2,518 |
| Mean (daily) | 0.000469 | 0.000342 | 0.000604 | 0.000461 | 0.000406 | 0.000255 | 0.000454 |
| StDev (daily) | 0.011125 | 0.014034 | 0.013467 | 0.011018 | 0.012672 | 0.012557 | 0.013961 |
| Skewness | -0.837898 | -1.015128 | -0.580441 | -0.978364 | -1.114062 | -1.769440 | -0.589995 |
| Autocorrelation | -0.145757 | -0.106853 | -0.132271 | -0.147274 | -0.096378 | -0.065043 | -0.120480 |
| Minimum | -0.127605 | -0.153278 | -0.130017 | -0.138418 | -0.147921 | -0.191182 | -0.150632 |
| Median | 0.000718 | 0.000984 | 0.001139 | 0.000754 | 0.000835 | 0.000846 | 0.000857 |
| Maximum | 0.089771 | 0.089804 | 0.095979 | 0.107643 | 0.101732 | 0.083040 | 0.124440 |
| Mean (annual) | 11.83% | 8.62% | 15.23% | 11.61% | 10.22% | 6.43% | 11.44% |
| StDev (annual) | 17.66% | 22.28% | 21.38% | 17.49% | 20.12% | 19.93% | 22.16% |
| CAGR | 12.56% | 9.00% | 16.45% | 12.31% | 10.77% | 6.64% | 12.12% |
| HPR | 226.12% | 136.56% | 357.97% | 218.90% | 177.77% | 90.06% | 213.70% |

Nasdaq had highest price appreciation.

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# Stock index returns



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Russel 2000 index had highest volatility.

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# Stock index returns

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| **Description** | **SPX** | **RTY** | **NDX** | **INDU** | **MID** | **DJUSRE** | **S5FINL** |
| *n* | 2,518 | 2,518 | 2,518 | 2,518 | 2,518 | 2,518 | 2,518 |
| Mean (daily) | 0.000469 | 0.000342 | 0.000604 | 0.000461 | 0.000406 | 0.000255 | 0.000454 |
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Major indexes had negative skewness.

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# Stock index returns

| Summary statistics of daily index returns for period 20091231 through 20192131 | | | | | | | |
|---|---|---|---|---|---|---|---|
| | S&P 500 | RUSS2000 | Nasdaq 100 | Dow | Midcap 400 | DJ RE | S&P Fin |
| *n* | 2,516 | 2,516 | 2,516 | 2,516 | 2,516 | 2,516 | 2,516 |
| Mean | 0.000483 | 0.000504 | 0.000711 | 0.000465 | 0.000528 | 0.000365 | 0.000498 |
| StDev | 0.009953 | 0.013373 | 0.010964 | 0.009288 | 0.011610 | 0.015936 | 0.016699 |
| Skewness | -0.315213 | -0.135942 | -0.245333 | -0.298776 | -0.283405 | 0.373483 | 0.446630 |
| Minimum | -0.068958 | -0.093317 | -0.063053 | -0.057061 | -0.086123 | -0.115888 | -0.121282 |
| Median | 0.000362 | 0.000554 | 0.000773 | 0.000374 | 0.000606 | 0.000382 | 0.000273 |
| Maximum | 0.068366 | 0.080660 | 0.063621 | 0.066116 | 0.071101 | 0.152206 | 0.163312 |
| CAGR | 12.93% | 13.54% | 19.63% | 12.42% | 14.24% | 9.64% | 13.37% |
| Volalitily | 15.80% | 21.23% | 17.41% | 14.74% | 18.43% | 25.30% | 26.51% |
| HPR | 236.70% | 255.43% | 498.47% | 221.85% | 277.74% | 150.62% | 249.93% |

Sector indexes had positive skewness.

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# Stock index returns

| Correlation matrix | | | | | | | |
|---|---|---|---|---|---|---|---|
| | SPX | RTY | NDX | INDU | MID | DJUSRE | S5FINL |
| SPX | 1 | | | | | | |
| RTY | 0.884 | 1 | | | | | |
| NDX | 0.931 | 0.797 | 1 | | | | |
| INDU | 0.965 | 0.857 | 0.838 | 1 | | | |
| MID | 0.923 | 0.968 | 0.803 | 0.909 | 1 | | |
| DJUSRE | 0.765 | 0.722 | 0.635 | 0.752 | 0.782 | 1 | |
| S5FINL | 0.873 | 0.841 | 0.698 | 0.904 | 0.889 | 0.686 | 1 |

For major indexes, strongest correlation is between S&P and Dow.

13

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For major indexes, strongest correlation is between S&P and Dow.

- Implies Dow futures market will not compete for hedging purposes.

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# Stock index returns

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For major indexes, weakest correlation is between S&P and Russell 2000.

- Implies Russell 2000 futures market can compete for hedging purposes.

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# Stock index futures

| Product name | Index | Volume | Open interest | Turnover |
|---|---|---|---|---|
| E-mini S&P 500 Futures | S&P | 1,757,340 | 2,205,411 | 79.7% |
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## Stock index futures

- Most active index futures in U.S. is E-mini S&P 500 contract.
  - Is 50 times index level.
  - Expires at open on third Friday of month.
  - Is cash-settled to special index level based on opening trade prices of each of S&P 500 stocks.
  - Weekly expirations have been launched but garner little trading volume.

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## Composition of stock indexes

- Value-weighted arithmetic indexes
  - S&P 500, Russell 2000, S&P 400, Nasdaq 100
- Price-weighted arithmetic index
  - Dow

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## Value-weighted index

- S&P 500 index:
  - Included 500 stocks for first time in 1957.
  - Initial divisor was computed using average share prices of index stocks during period 1941-3.
    - Base index level was set equal to 10.
    - Current index level is about 3326.50.
    - Price appreciation of about 33,165%.

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## Price-weighted index

- DJIA 30:
  - Began with 12 “blue-chip” stocks on May 26, 1896.
    - Average price on that day was 40.94.
  - Increased to 20 stocks in 1916.
  - Increased to 30 stocks in 1928.

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## Cost of carry model

- Have two versions of cost of carry model
  - Income is constant continuous rate.
  - Income is discrete payments.
- Which one should be used for stock index futures?

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## S&P 500 dividends

- Support file: S&P 500 index dividends.xlsx
  - Downloaded daily data from Datastream for period 19871231 through 20191231.
    - Eliminated non-trading days.

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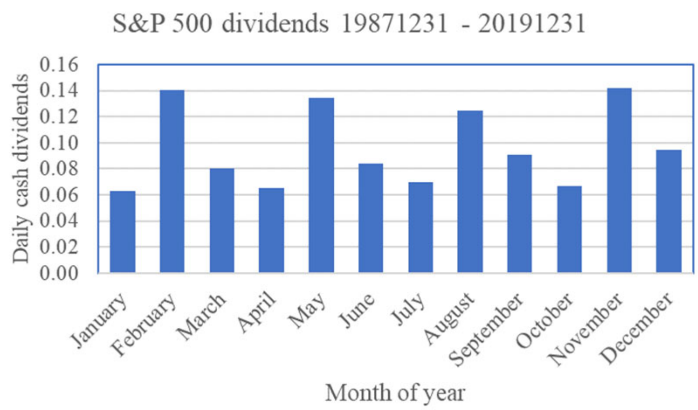
# Dividend patterns

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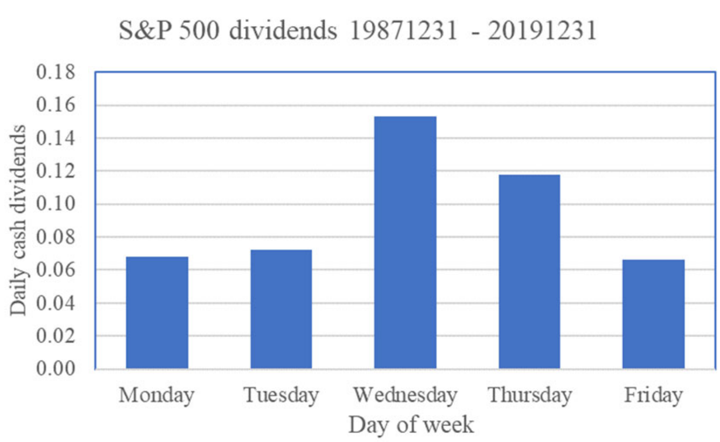
# Dividend patterns

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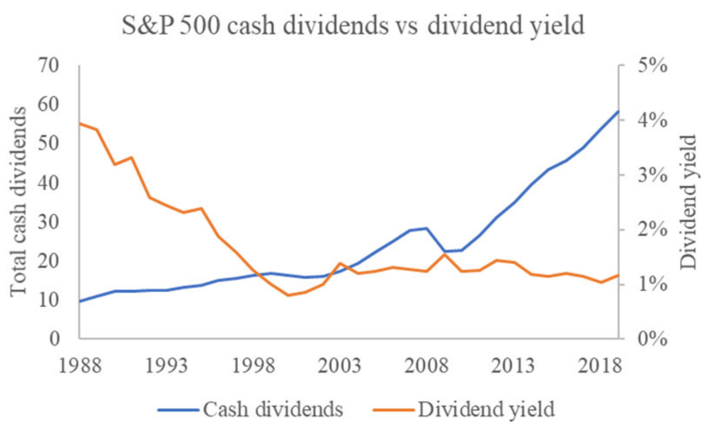
# Dividend patterns

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# Stock index futures valuation

- Need to handled dividends as discrete flows.
- Net cost of carry relation for stock index futures is:

$$F = Se^{rT} - \sum_{i=1}^{n} D_i e^{r(T-t_i)}$$

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# Stock index arbitrage

- Trading costs for index arbitrageurs are about 1.50 index points.
  - Will execute an arbitrage if

$$F < Se^{rT} - \sum_{i=1}^{n} D_i e^{r(T-t_i)} - 1.50$$

  - Buy futures, sell index portfolio, invest proceeds in T-bills.

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# Stock index arbitrage

- Trading costs for index arbitrageurs are about 1.50 index points.
  - Will execute an arbitrage if

$$F > Se^{rT} - \sum_{i=1}^{n} D_i e^{r(T-t_i)} - 1.50$$

  - Sell futures, borrow, and buy index portfolio.

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# Stock index arbitrage

- How efficient is index arbitrage?
  - On a minute-by-minute basis on 8/29/03, computed

$$\text{Mispricing} = F - \left[ Se^{rT} - \sum_{i=1}^{n} D_i e^{r(T-t_i)} \right]$$

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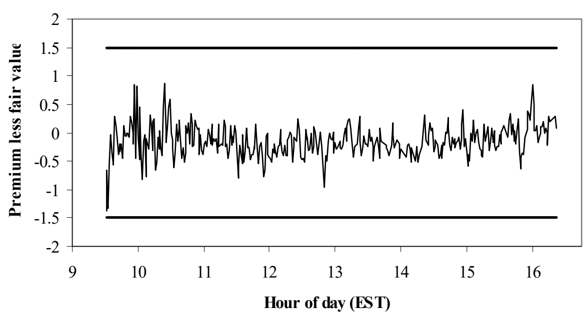
# Stock index arbitrage

- How efficient is index arbitrage?
  - Mispricing and arbitrage bounds on 8/29/03

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## Stock index arbitrage

- Buying or selling of index stocks must be *simultaneous*.
  - Requires use of computer-generated orders.
  - Referred to as *basket trading* or *program trading*.

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## Hedging with index futures

- Manage stock portfolio risk by buying and selling index futures contracts.
  - Assume objective is to minimize risk of hedged portfolio return (subject to given level of return) using available futures contracts.

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## Hedging with index futures

- Elements of least-risk hedging:
  - Identify least-risk hedge ratio.
  - Show equivalence of hedge ratio to OLS regression slope coefficient.
  - Discuss estimation issues.
  - Generalize model to multiple sources of risk.

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## Hedging with index futures

- Notation:

$V_0 =$ initial value of portfolio to be hedged

$\tilde{V}_T =$ uncertain portfolio value at time $T$

$\tilde{R}_V = \ln\left(\frac{\tilde{V}_T}{V_0}\right) =$ return on portfolio to be hedged

$F_0 =$ initial futures price

$\tilde{F}_T =$ uncertain futures price at time $T$

$\tilde{R}_F = \ln\left(\frac{\tilde{F}_T}{F_0}\right) =$ return on futures

$n_F =$ number of futures contracts bought (+) or sold (–)

$\tilde{R}_H =$ return on hedged portfolios

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## Hedging with index futures

- Hedged portfolio return is:

$$\tilde{R}_H = \tilde{R}_V + n_F \tilde{R}_F$$

- Risk of hedged portfolio return is:

$$\sigma_H^2 = \sigma_V^2 + n_F^2 \sigma_F^2 + 2 n_F \sigma_{V,F}$$

- Least-risk hedge is determined by:

$$\frac{d\sigma_H^2}{n_F} = 2 n_F^* \sigma_F^2 + 2\sigma_{V,F} = 0$$

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## Hedging with index futures

- Least-risk hedge ratio is:

$$n_F^* = -\frac{\sigma_{V,F}}{\sigma_F^2} = -\rho_{V,F}\left(\frac{\sigma_V}{\sigma_F}\right)$$

- Least-risk hedge ratio if portfolio being hedged underlies futures (e.g., S&P 500)?

$$n_F^* = -\rho_{V,F}\left(\frac{\sigma_V}{\sigma_F}\right) = -1$$

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## Hedging with index futures

- Illustration: Find least-risk hedge ratio for stock portfolio.
  - Support file: Stock portfolio hedge.xlsx
    - Contains daily values of stock portfolio and index stock futures prices.
    - Computes hedge ratio:
      - Analytically
      - Using regression
      - Using SOLVER

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## Hedging with index futures

- Step 1: Identify appropriate futures contract(s).
  - Since no futures are written on stock portfolio, identify closest, liquid substitute.
    - Examine correlation between stock portfolio and available index futures. Generally want contract with highest correlation.
    - Depth and liquidity of index futures is also important.
      - Depth ensures small price impact.
      - Liquidity (i.e., small bid/ask spread) ensures fast and cost-efficient trading.

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# Hedging with index futures

- Step 2: Collect historical time series.
  - Daily stock portfolio values
  - Index futures prices
    - Index futures price contract denomination is 50.

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# Hedging with index futures

- Step 3: Estimate parameters of minimum risk hedge.
  - Compute:
    - Standard deviation of stock portfolio returns
    - Standard deviation of futures returns
    - Correlation between stock portfolio and futures returns

| Summary statistics for returns | Stock portfolio return | Index futures return |
|---|---|---|
| *n* | 294 | 294 |
| Mean | 0.000405 | 0.000333 |
| StDev | 0.005151 | 0.005706 |
| Skewness | 0.009638 | 0.075221 |
| Correlation | 0.989811 | |

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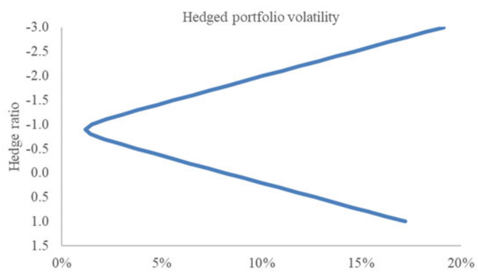
# Hedging with index futures

- Step 4: Compute least-risk hedge ratio.

$$n_F^* = -.989811\left(\frac{.005151}{.005711}\right) = -.893543$$

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# Hedging with index futures

- Step 5: Determine least-risk number of futures.
  - Hedge ratio is number of futures to sell per unit of stock portfolio.
    - Need to adjust by dollar value of portfolio and dollar value of stock index futures.

$$\begin{aligned} n_F^* &= -.89354\left(\frac{32,671,455}{1,152.50 \times 50}\right) \\ &= -.89354\left(\frac{32,671,455}{1,152.50 \times 50}\right) \\ &= -506.61 \approx -507 \end{aligned}$$

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## Regression approach

- OLS regression provides alternative means for risk measurement.
- Consider regression of portfolio return on futures return.

$$\tilde{R}_V = \alpha_0 + \alpha_1 \tilde{R}_F + \tilde{\varepsilon}$$

- Hedge portfolio return may be written as:

$$\begin{aligned}\tilde{R}_H &= \tilde{R}_V + n_F \tilde{R}_F \\ &= \alpha_0 + \alpha_1 \tilde{R}_F + \tilde{\varepsilon} + n_F \tilde{R}_F \\ &= \alpha_0 + (\alpha_1 + n_F)\tilde{R}_F + \tilde{\varepsilon}\end{aligned}$$

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## Regression approach

- Variance of hedged portfolio return is:

$$\begin{aligned}Var(\tilde{R}_H) &= Var\left[\alpha_0 + (\alpha_1 + n_F)\tilde{R}_F + \tilde{\varepsilon}\right] \\ &= Var\left[(\alpha_1 + n_F)\tilde{R}_F\right] + Var(\tilde{\varepsilon}) + 2Cov\left((\alpha_1 + n_F)\tilde{R}_F, \tilde{\varepsilon}\right) \\ &= Var\left[(\alpha_1 + n_F)\tilde{R}_F\right] + Var(\tilde{\varepsilon})\end{aligned}$$

- Where is hedged portfolio risk minimized?

$$n_F^* = -\alpha_1$$

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## Regression approach

- Illustration: Find risk-minimizing hedge for portfolio of stocks.
  - Support file: Stock portfolio hedge.xlsx

Same risk-minimizing hedge ratio.

| Regression Statistics | |
|---|---|
| Multiple R | 0.9898 |
| R Square | 0.9797 |
| Adjusted R Square | 0.9797 |
| Standard Error | 0.0007 |
| Observations | 294 |

| | *Coeff* | *StErr* | *t Stat* |
|---|---|---|---|
| Intercept | 0.00011 | 0.00004 | 2.50 |
| Futures return | $\boxed{0.89354}$ | 0.00752 | 118.79 |

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## Regression approach

- Illustration: Find risk-minimizing hedge for portfolio of stocks.
  - Support file: Stock portfolio hedge.xlsx

Adjusted R Squared is measure of *hedging effectiveness*.

Percent of portfolio return variance explained by futures returns.

| Regression Statistics | |
|---|---|
| Multiple R | 0.9898 |
| R Square | 0.9797 |
| Adjusted R Square | $\boxed{0.9797}$ |
| Standard Error | 0.0007 |
| Observations | 294 |

| | *Coeff* | *StErr* | *t Stat* |
|---|---|---|---|
| Intercept | 0.00011 | 0.00004 | 2.50 |
| Futures return | 0.89354 | 0.00752 | 118.79 |

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## Estimation issues

- Return interval must be selected (e.g., daily, weekly, monthly, etc. returns).
  - Higher frequency implies more information (good) but also more noise (bad).
  - Prices for cash and futures must be simultaneous.

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## Estimation issues

- Illustration: Suppose we hold S&P 500 index portfolio and want to hedge market risk using S&P 500 futures.
  - Already know least-risk hedge ratio is –1.

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## Estimation issues

- Suppose we use historical data to estimate hedge.
- S&P 500 futures data during 1997
  - 1997 had 254 trading days, which creates:
    - 253 daily returns
    - 52 weekly returns
    - 26 biweekly returns
  - For weekly and biweekly returns, Wednesday closing prices are used.

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## Estimation issues

- Use returns of nearby futures contract.
  - When switching contract months, care must be taken to splice price change series correctly.
- Support file: S&P 500 hedge.xlsx

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# Estimation issues

- Daily, weekly, and bi-weekly regressions using S&P 500 futures data during 1997.

**Hedge regressions using different return intervals**

| | $\alpha_0$ | $t(\alpha_0)$ | $\alpha_1$ | $t(\alpha_1)$ | Adjusted R-squared |
|---|---|---|---|---|---|
| Daily | 0.0003 | 1.65 | 0.8764 | 70.07 | 0.9514 |
| Weekly | 0.0009 | 2.29 | 0.9558 | 55.64 | 0.9841 |
| Bi-weekly | 0.0016 | 2.61 | 0.9884 | 43.40 | 0.9874 |

Measurement error can be large.
Correct answer is 1.

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# Estimation issues

- CAVEAT: Parameters are estimated using past data, but we are interested in future.
  - Must have reason to believe relation is stationary.

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## Hedging multiple risks

- OLS approach can be generalized to handle multiple sources of risk.

$$\tilde{R}_V = \alpha_0 + \alpha_1 \tilde{R}_{F,1} + \alpha_2 \tilde{R}_{F,2} + \ldots + \alpha_n \tilde{R}_{F,n} + \tilde{\varepsilon}$$

- Set hedge ratios for all or just selected risks.

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## Hedging multiple risks

- Illustration: Hedge oil price risk of fund that invests primarily in oil stocks.

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# Hedging multiple risks

- Step 1: Identify portfolio risk exposures and find futures to proxy for each.
  - SP: S&P 500 futures (stock market risk)
  - CL: crude oil futures (petroleum price risk)
  - HO: heating oil futures (processed petroleum price risk)
  - HU: unleaded gas futures (processed petroleum price risk)

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# Hedging multiple risks

- Step 2: Collect historical return data.
  - Support file: Oil hedge.xlsx

| | Portfolio value | Portfolio | Futures prices | | | | Futures returns | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Day | ($ millions) | return | SP | CL | HO | HU | SP | CL | HO | HU |
| 1 | 769.26 | | 1100.00 | 40.00 | 0.6900 | 0.7500 | | | | |
| 2 | 776.57 | 0.00946 | 1099.00 | 40.39 | 0.6969 | 0.7616 | -0.00091 | 0.00970 | 0.00995 | 0.01535 |
| 3 | 782.45 | 0.00754 | 1100.55 | 40.50 | 0.7051 | 0.7709 | 0.00141 | 0.00272 | 0.01170 | 0.01214 |
| 4 | 791.62 | 0.01165 | 1101.10 | 41.17 | 0.7236 | 0.7878 | 0.00050 | 0.01641 | 0.02590 | 0.02169 |
| 5 | 791.44 | -0.00023 | 1100.90 | 41.25 | 0.7213 | 0.7865 | -0.00018 | 0.00194 | -0.00318 | -0.00165 |
| 6 | 789.32 | -0.00268 | 1104.30 | 41.15 | 0.7265 | 0.7882 | 0.00308 | -0.00243 | 0.00718 | 0.00216 |
| 7 | 793.38 | 0.00513 | 1109.05 | 40.50 | 0.7124 | 0.7670 | 0.00429 | -0.01592 | -0.01960 | -0.02727 |
| 8 | 789.23 | -0.00524 | 1108.40 | 40.68 | 0.7290 | 0.7764 | -0.00059 | 0.00443 | 0.02303 | 0.01218 |
| 9 | 786.04 | -0.00405 | 1107.25 | 40.16 | 0.7255 | 0.7673 | -0.00104 | -0.01287 | -0.00481 | -0.01179 |
| 10 | 784.78 | -0.00160 | 1106.65 | 40.34 | 0.7339 | 0.7690 | -0.00054 | 0.00447 | 0.01151 | 0.00221 |

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# Hedging multiple risks

- Step 3: Get to know properties of data.

SP volatility low
during sample period.

CL and processed
products
are more normal.

| Summary statistics | | | | |
|---|---|---|---|---|
| | Futures returns | | | |
| | SP | CL | HO | HU |
| *n* | 252 | 252 | 252 | 252 |
| Mean | -0.000029 | 0.000483 | 0.000773 | 0.000609 |
| StDev | 0.002798 | 0.006899 | 0.011017 | 0.011879 |
| Skewness | -0.473383 | -0.162212 | -0.207878 | -0.265877 |
| CAGR | -0.73% | 12.95% | 21.49% | 16.60% |
| Volatility | 4.44% | 10.95% | 17.49% | 18.86% |

| Correlations | | | | |
|---|---|---|---|---|
| | SP | CL | HO | HU |
| SP | 1 | | | |
| CL | -0.191 | 1 | | |
| HO | -0.135 | 0.773 | 1 | |
| HU | -0.079 | 0.718 | 0.662 | 1 |

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# Hedging multiple risks

- Step 3: Get to know properties of data.

SP volatility low
during sample period.

CL and processed products
are more normal.

CL less volatility than HO
or HU.

| Summary statistics | | | | |
|---|---|---|---|---|
| | Futures returns | | | |
| | SP | CL | HO | HU |
| *n* | 252 | 252 | 252 | 252 |
| Mean | -0.000029 | 0.000483 | 0.000773 | 0.000609 |
| StDev | 0.002798 | 0.006899 | 0.011017 | 0.011879 |
| Skewness | -0.473383 | -0.162212 | -0.207878 | -0.265877 |
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| Correlations | | | | |
|---|---|---|---|---|
| | SP | CL | HO | HU |
| SP | 1 | | | |
| CL | -0.191 | 1 | | |
| HO | -0.135 | 0.773 | 1 | |
| HU | -0.079 | 0.718 | 0.662 | 1 |

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# Hedging multiple risks

- Step 3: Get to know properties of data.

What does this say about oil oil stock exposure?

Are stocks:
1) Oil exploration
2) Oil refining, or
3) Oil production distribution?

**Summary statistics**

| | Futures returns | | | |
|---|---|---|---|---|
| | SP | CL | HO | HU |
| *n* | 252 | 252 | 252 | 252 |
| Mean | -0.000029 | 0.000483 | 0.000773 | 0.000609 |
| StDev | 0.002798 | 0.006899 | 0.011017 | 0.011879 |
| Skewness | -0.473383 | -0.162212 | -0.207878 | -0.265877 |
| CAGR | -0.73% | 12.95% | 21.49% | 16.60% |
| Volatility | 4.44% | 10.95% | 17.49% | 18.86% |

**Correlations**

| | SP | CL | HO | HU |
|---|---|---|---|---|
| SP | 1 | | | |
| CL | -0.191 | 1 | | |
| HO | -0.135 | 0.773 | 1 | |
| HU | -0.079 | 0.718 | 0.662 | 1 |

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# Hedging multiple risks

- Step 4: Run OLS regression.

$$\tilde{R}_V = \alpha_0 + \alpha_1 \tilde{R}_{F,1} + \alpha_2 \tilde{R}_{F,2} + \ldots + \alpha_n \tilde{R}_{F,n} + \tilde{\varepsilon}$$

Hedge is not (will not be) particularly effective.

**Regression Statistics**

| | |
|---|---|
| Multiple R | 0.5826 |
| R Square | 0.3394 |
| Adjusted R Square | 0.3287 |
| Standard Error | 0.0060 |
| Observations | 252 |

| | *Coeff* | *StErr* | *t Stat* |
|---|---|---|---|
| Intercept | 0.00002 | 0.00038 | 0.05 |
| SP | 1.55502 | 0.13904 | 11.18 |
| CL | 0.21055 | 0.09773 | 2.15 |
| HO | -0.03085 | 0.05610 | -0.55 |
| HU | -0.00062 | 0.04764 | -0.01 |

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# Hedging multiple risks

- Step 4: Run OLS regression.

$$\tilde{R}_V = \alpha_0 + \alpha_1 \tilde{R}_{F,1} + \alpha_2 \tilde{R}_{F,2} + \ldots + \alpha_n \tilde{R}_{F,n} + \tilde{\varepsilon}$$

HO and HU returns have little effect on portfolio return.

Drop HO and HU and re-run regression.

| Regression Statistics | |
|---|---|
| Multiple R | 0.5826 |
| R Square | 0.3394 |
| Adjusted R Square | 0.3287 |
| Standard Error | 0.0060 |
| Observations | 252 |

| | *Coeff* | *StErr* | *t Stat* |
|---|---|---|---|
| Intercept | 0.00002 | 0.00038 | 0.05 |
| SP | 1.55502 | 0.13904 | 11.18 |
| CL | 0.21055 | 0.09773 | 2.15 |
| HO | -0.03085 | 0.05610 | -0.55 |
| HU | -0.00062 | 0.04764 | -0.01 |

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# Hedging multiple risks

- Step 5: Drop heating oil and crude oil and re-run regression.

Adjusted R-squared is increased.

| Regression Statistics | |
|---|---|
| Multiple R | 0.5818 |
| R Square | 0.3385 |
| Adjusted R Square | 0.3332 |
| Standard Error | 0.0060 |
| Observations | 252 |

| | *Coeff* | *StErr* | *t Stat* |
|---|---|---|---|
| Intercept | 0.00001 | 0.00038 | 0.04 |
| SP | 1.55328 | 0.13808 | 11.25 |
| CL | 0.17158 | 0.05600 | 3.06 |

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# Hedging multiple risks

- Step 5: Drop heating oil and crude oil and re-run regression.

Adjusted R-squared is increased.

Both SP and CL returns explain portfolio returns.

| Regression Statistics | |
|---|---|
| Multiple R | 0.5818 |
| R Square | 0.3385 |
| Adjusted R Square | 0.3332 |
| Standard Error | 0.0060 |
| Observations | 252 |

| | *Coeff* | *StErr* | *t Stat* |
|---|---|---|---|
| Intercept | 0.00001 | 0.00038 | 0.04 |
| SP | 1.55328 | 0.13808 | 11.25 |
| CL | 0.17158 | 0.05600 | 3.06 |

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# Hedging multiple risks

- Step 6: Find number of crude oil futures to sell.

$$n_F^* = -.17158\left(\frac{779{,}520{,}000}{45.18 \times 1{,}000}\right) = -2{,}960.36 \approx -2{,}960$$

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# Estimation issues

- Missing variable bias: Regress on crude oil futures return only.

| Regression Statistics | |
|---|---|
| Multiple R | 0.0482 |
| R Square | 0.0023 |
| Adjusted R Square | -0.0017 |
| Standard Error | 0.0074 |
| Observations | 252 |

| | *Coeff* | *StErr* | *t Stat* |
|---|---|---|---|
| Intercept | 0.00003 | 0.00047 | 0.06 |
| CL | 0.05136 | 0.06738 | 0.76 |

$$n_F^* = -.05136\left(\frac{779{,}520{,}000}{45.18 \times 1{,}000}\right) = -886.22 \approx -886$$

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# Estimation issues

- Missing variable bias: Recall return correlations.

| Correlations | SP | CL | HO | HU |
|---|---|---|---|---|
| SP | 1 | | | |
| CL | -0.191 | 1 | | |
| HO | -0.135 | 0.773 | 1 | |
| HU | -0.079 | 0.718 | 0.662 | 1 |

Absence of SP means that CL, to some degree, also picks up effect of CL because SP and CL are correlated.

Negative correlation implies downward bias.

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# Stock index options

- Index options began trading in March 1983.
  - CBOE introduced options on CBOE 100
    - Called “OEX” options
    - Index later became “S&P 100”
- Index futures options began in March 1983.
  - CME introduced options on S&P 500 futures.

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# Stock index options

**Options and Futures Volume by Exchange — February 6, 2020**

Reports are available for up to two (2) years. Batch processing information can be found here.

Options

| Exchange | Equity | | Index/Other | | Debt | | Exchange Total | |
|---|---|---|---|---|---|---|---|---|
| | Volume | Market Share | Volume | Market Share | Volume | Market Share | Volume | Market Share |
| AMEX | 1,901,498 | 8.42% | 0 | 0.00% | 0 | 0.00% | 1,901,498 | 7.75% |
| ARCA | 1,966,771 | 8.71% | 0 | 0.00% | 0 | 0.00% | 1,966,771 | 8.02% |
| BATS | 2,101,429 | 9.30% | 836 | 0.04% | 0 | 0.00% | 2,102,265 | 8.57% |
| BOX | 717,422 | 3.18% | 0 | 0.00% | 0 | 0.00% | 717,422 | 2.93% |
| C2 | 809,935 | 3.59% | 6,289 | 0.33% | 0 | 0.00% | 816,224 | 3.33% |
| CBOE | 3,247,798 | 14.38% | 1,908,389 | 98.81% | 0 | 0.00% | 5,156,187 | 21.03% |
| EDGX | 828,137 | 3.67% | 513 | 0.03% | 0 | 0.00% | 828,650 | 3.38% |
| EMLD | 171,564 | 0.76% | 0 | 0.00% | 0 | 0.00% | 171,564 | 0.70% |
| GEM | 1,023,713 | 4.53% | 300 | 0.02% | 0 | 0.00% | 1,024,013 | 4.18% |
| ISE | 1,837,807 | 8.14% | 9,984 | 0.52% | 0 | 0.00% | 1,847,791 | 7.54% |
| MCRY | 90,808 | 0.40% | 0 | 0.00% | 0 | 0.00% | 90,808 | 0.37% |
| MIAX | 1,038,088 | 4.60% | 1 | 0.00% | 0 | 0.00% | 1,038,089 | 4.23% |
| MPRL | 1,171,263 | 5.18% | 0 | 0.00% | 0 | 0.00% | 1,171,263 | 4.78% |
| NOBO | 51,299 | 0.23% | 0 | 0.00% | 0 | 0.00% | 51,299 | 0.21% |
| NSDQ | 2,123,738 | 9.40% | 0 | 0.00% | 0 | 0.00% | 2,123,738 | 8.66% |
| PHLX | 3,508,899 | 15.53% | 5,090 | 0.26% | 0 | 0.00% | 3,513,989 | 14.33% |
| OCC Totals | 22,590,169 | 100.00% | 1,931,402 | 100.00% | 0 | 0.00% | 24,521,571 | 100.00% |

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## Stock index futures options

| Product name | Index | Volume | Open interest | Turnover |
|---|---|---|---|---|
| E-mini S&P 500 Options | S&P | 133,176 | 2,204,523 | 6.04% |
| E-mini S&P 500 Weekly Options - Week 3 | S&P | 385,483 | 2,180,041 | 17.68% |
| E-mini S&P 500 EOM Options | S&P | 70,882 | 822,073 | 8.62% |
| E-mini S&P 500 Monday Weekly Options - Week 2 | S&P | 307,734 | 466,045 | 66.03% |
| E-mini S&P 500 Tuesday Weekly Options - Week 2 | S&P | 176,320 | 258,164 | 68.30% |
| E-mini S&P 500 Weekly Options - Week 4 | S&P | 38,395 | 227,094 | 16.91% |
| E-mini S&P 500 Weekly Options - Week 1 | S&P | 16,923 | 159,540 | 10.61% |
| E-mini Nasdaq-100 Options | Nasdaq | 12,738 | 138,975 | 9.17% |
| E-mini S&P 500 Wednesday Weekly Options - Week 3 | S&P | 43,758 | 121,504 | 36.01% |
| E-mini S&P 500 Weekly Options - Week 2 | S&P | 493,644 | 120,096 | 411.04% |
| E-mini S&P 500 Thursday Weekly Options - Week 3 | S&P | 29,676 | 83,101 | 35.71% |
| Weekly Options on Micro E-mini Standard and Poors 500 Stock Price Index Futures - Week 3 | S&P | 5,975 | 71,884 | 8.31% |
| E-mini S&P 500 Monday Weekly Options - Week 3 | S&P | 25,730 | 54,460 | 47.25% |

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## Stock index option valuation

- For European-style options written on stock index, reduce index level by present value of promised dividends.

$$PVD = \sum_{i=1}^{n} De^{-r(T-t_i)}$$

$$S^x = S - PVD$$

- Valuation equations are:

Put-call parity: $c - p = S^x - Xe^{-rT}$

BSM call option formula: $c = S^x N\left(d_1\right) - Xe^{-rT} N\left(d_2\right)$

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## Stock index futures option valuation

- For European-style options written on stock index futures, valuation equations are:

$$\text{Put-call parity: } c - p = e^{-rT}\left(F - X\right)$$

$$\text{BSM call option formula: } c = e^{-rT}\left(FN\left(d_1\right) - XN\left(d_2\right)\right)$$

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## Risk management strategies

- Insure stock portfolios.
- Create protected equity notes.
- Create enhanced buffered notes.

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## Portfolio insurance

- *Portfolio insurance* refers to insuring value of portfolio of assets, most commonly stocks.
  - One of most important financial innovations of 1980s.

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## Portfolio insurance

- Two types of portfolio insurance:
  - *Passive portfolio insurance*: buy index put option while holding stock portfolio
    - Index puts began trading in 1983.
  - *Dynamic portfolio insurance*: dynamically rebalance portfolio of stocks and T-bills
    - Idea conceived by Hayne Leland in 1976
    - LOR had more than $60B under insurance programs by October 1987 market crash.
    - Crash killed business.

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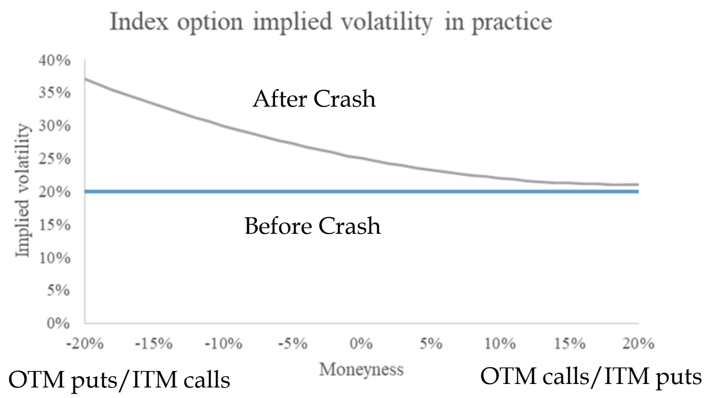
# S&P 500 index option idiosyncrasies

- S&P 500 index option implied volatility before and after October 1987 stock market crash.

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# Passive portfolio insurance

- *Passive portfolio insurance* involves buying index put against a position in a stock portfolio. Value of position is:

$$V = Se^{-\delta T} + p$$

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# Passive portfolio insurance

- Cost of insurance (or value of index put) is:

$$p = Xe^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

where

$$d_1 = \frac{\ln(S/X) + (r - \delta + .5\sigma^2)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

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# Passive portfolio insurance

- Illustration:
  - Assume index portfolio has
    - Value = $50 million
    - Volatility rate = 20%
    - Dividend yield rate = 1.5%
  - Assume
    - S&P 500 index level = 1,500
    - Risk-free rate = 6%
  - Want to ensure portfolio value is at least $50 million in one year.
  - Support file: Passive PI.xlsx

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## Passive portfolio insurance

- Illustration:
  - Step 1: Identify number of index units today and at end of one year.
    - Denomination of put is 100.
    - Number of units today is

$$n = \frac{50,000,000}{1,500(100)} = 333.333$$

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## Passive portfolio insurance

- Illustration:
  - With re-investment of dividend yield, number of index units in one year will be:

$$n = 333.33e^{.015(1)} = 338.371$$

  - Need to buy 338.371 index puts.

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# Passive portfolio insurance

- Illustration:
  - Step 2: Identify exercise price of put.

$$X = \frac{50,000,000}{338.371(100)} = 1,477.67$$

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# Passive portfolio insurance

- Illustration:
  - Step 3: Identify cost of insurance.
    - Cost of each put is \$76.34 per index unit (according to BSM formula).
    - Total cost is \$2,583,000.

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# Passive portfolio insurance

Examine values at expiration.

| | Portfolio values at expiration | | |
|---|---|---|---|
| Index level | Stock portfolio value | Value of put | Insured portfolio value |
| 500 | 16,918,551 | 33,081,449 | 50,000,000 |
| 600 | 20,302,261 | 29,697,739 | 50,000,000 |
| 700 | 23,685,972 | 26,314,028 | 50,000,000 |
| 800 | 27,069,682 | 22,930,318 | 50,000,000 |
| 900 | 30,453,392 | 19,546,608 | 50,000,000 |
| 1,000 | 33,837,102 | 16,162,898 | 50,000,000 |
| 1,100 | 37,220,812 | 12,779,188 | 50,000,000 |
| 1,200 | 40,604,523 | 9,395,477 | 50,000,000 |
| 1,300 | 43,988,233 | 6,011,767 | 50,000,000 |
| 1,400 | 47,371,943 | 2,628,057 | 50,000,000 |
| 1,500 | 50,755,653 | 0 | 50,755,653 |
| 1,600 | 54,139,363 | 0 | 54,139,363 |
| 1,700 | 57,523,074 | 0 | 57,523,074 |
| 1,800 | 60,906,784 | 0 | 60,906,784 |
| 1,900 | 64,290,494 | 0 | 64,290,494 |
| 2,000 | 67,674,204 | 0 | 67,674,204 |
| 2,100 | 71,057,915 | 0 | 71,057,915 |
| 2,200 | 74,441,625 | 0 | 74,441,625 |
| 2,300 | 77,825,335 | 0 | 77,825,335 |
| 2,400 | 81,209,045 | 0 | 81,209,045 |
| 2,500 | 84,592,755 | 0 | 84,592,755 |

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# Passive portfolio insurance

Examine values at expiration.
- Protected on downside.

| | Portfolio values at expiration | | |
|---|---|---|---|
| Index level | Stock portfolio value | Value of put | Insured portfolio value |
| 500 | 16,918,551 | 33,081,449 | 50,000,000 |
| 600 | 20,302,261 | 29,697,739 | 50,000,000 |
| 700 | 23,685,972 | 26,314,028 | 50,000,000 |
| 800 | 27,069,682 | 22,930,318 | 50,000,000 |
| 900 | 30,453,392 | 19,546,608 | 50,000,000 |
| 1,000 | 33,837,102 | 16,162,898 | 50,000,000 |
| 1,100 | 37,220,812 | 12,779,188 | 50,000,000 |
| 1,200 | 40,604,523 | 9,395,477 | 50,000,000 |
| 1,300 | 43,988,233 | 6,011,767 | 50,000,000 |
| 1,400 | 47,371,943 | 2,628,057 | 50,000,000 |
| 1,500 | 50,755,653 | 0 | 50,755,653 |
| 1,600 | 54,139,363 | 0 | 54,139,363 |
| 1,700 | 57,523,074 | 0 | 57,523,074 |
| 1,800 | 60,906,784 | 0 | 60,906,784 |
| 1,900 | 64,290,494 | 0 | 64,290,494 |
| 2,000 | 67,674,204 | 0 | 67,674,204 |
| 2,100 | 71,057,915 | 0 | 71,057,915 |
| 2,200 | 74,441,625 | 0 | 74,441,625 |
| 2,300 | 77,825,335 | 0 | 77,825,335 |
| 2,400 | 81,209,045 | 0 | 81,209,045 |
| 2,500 | 84,592,755 | 0 | 84,592,755 |

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# Passive portfolio insurance

Examine values at expiration.

- Retain upside.

| Portfolio values at expiration | | | |
|---|---|---|---|
| Index level | Stock portfolio value | Value of put | Insured portfolio value |
| 500 | 16,918,551 | 33,081,449 | 50,000,000 |
| 600 | 20,302,261 | 29,697,739 | 50,000,000 |
| 700 | 23,685,972 | 26,314,028 | 50,000,000 |
| 800 | 27,069,682 | 22,930,318 | 50,000,000 |
| 900 | 30,453,392 | 19,546,608 | 50,000,000 |
| 1,000 | 33,837,102 | 16,162,898 | 50,000,000 |
| 1,100 | 37,220,812 | 12,779,188 | 50,000,000 |
| 1,200 | 40,604,523 | 9,395,477 | 50,000,000 |
| 1,300 | 43,988,233 | 6,011,767 | 50,000,000 |
| 1,400 | 47,371,943 | 2,628,057 | 50,000,000 |
| 1,500 | 50,755,653 | 0 | 50,755,653 |
| 1,600 | 54,139,363 | 0 | 54,139,363 |
| 1,700 | 57,523,074 | 0 | 57,523,074 |
| 1,800 | 60,906,784 | 0 | 60,906,784 |
| 1,900 | 64,290,494 | 0 | 64,290,494 |
| 2,000 | 67,674,204 | 0 | 67,674,204 |
| 2,100 | 71,057,915 | 0 | 71,057,915 |
| 2,200 | 74,441,625 | 0 | 74,441,625 |
| 2,300 | 77,825,335 | 0 | 77,825,335 |
| 2,400 | 81,209,045 | 0 | 81,209,045 |
| 2,500 | 84,592,755 | 0 | 84,592,755 |

85

# Passive portfolio insurance

Examine initial values.

| Portfolio values with one year to expiration | | | |
|---|---|---|---|
| Index level | Stock portfolio value | Value of put | Insured portfolio value |
| 500 | 16,666,667 | 30,421,560 | 47,088,227 |
| 600 | 20,000,000 | 27,088,239 | 47,088,239 |
| 700 | 23,333,333 | 23,755,263 | 47,088,596 |
| 800 | 26,666,667 | 20,426,203 | 47,092,869 |
| 900 | 30,000,000 | 17,119,553 | 47,119,553 |
| 1,000 | 33,333,333 | 13,889,916 | 47,223,250 |
| 1,100 | 36,666,667 | 10,839,846 | 47,506,513 |
| 1,200 | 40,000,000 | 8,100,014 | 48,100,014 |
| 1,300 | 43,333,333 | 5,783,980 | 49,117,313 |
| 1,400 | 46,666,667 | 3,948,601 | 50,615,267 |
| 1,500 | 50,000,000 | 2,583,000 | 52,583,000 |
| 1,600 | 53,333,333 | 1,624,612 | 54,957,946 |
| 1,700 | 56,666,667 | 986,331 | 57,652,998 |
| 1,800 | 60,000,000 | 580,381 | 60,580,381 |
| 1,900 | 63,333,333 | 332,300 | 63,665,633 |
| 2,000 | 66,666,667 | 185,807 | 66,852,474 |
| 2,100 | 70,000,000 | 101,801 | 70,101,801 |
| 2,200 | 73,333,333 | 54,814 | 73,388,147 |
| 2,300 | 76,666,667 | 29,082 | 76,695,749 |
| 2,400 | 80,000,000 | 15,240 | 80,015,240 |
| 2,500 | 83,333,333 | 7,904 | 83,341,237 |

86

# Passive portfolio insurance

Examine initial values.

If index level falls to 500 immediately, portfolio value is 47,088,227.

| Portfolio values with one year to expiration | | | |
|---|---|---|---|
| Index level | Stock portfolio value | Value of put | Insured portfolio value |
| 500 | 16,666,667 | 30,421,560 | 47,088,227 |
| 600 | 20,000,000 | 27,088,239 | 47,088,239 |
| 700 | 23,333,333 | 23,755,263 | 47,088,596 |
| 800 | 26,666,667 | 20,426,203 | 47,092,869 |
| 900 | 30,000,000 | 17,119,553 | 47,119,553 |
| 1,000 | 33,333,333 | 13,889,916 | 47,223,250 |
| 1,100 | 36,666,667 | 10,839,846 | 47,506,513 |
| 1,200 | 40,000,000 | 8,100,014 | 48,100,014 |
| 1,300 | 43,333,333 | 5,783,980 | 49,117,313 |
| 1,400 | 46,666,667 | 3,948,601 | 50,615,267 |
| 1,500 | 50,000,000 | 2,583,000 | 52,583,000 |
| 1,600 | 53,333,333 | 1,624,612 | 54,957,946 |
| 1,700 | 56,666,667 | 986,331 | 57,652,998 |
| 1,800 | 60,000,000 | 580,381 | 60,580,381 |
| 1,900 | 63,333,333 | 332,300 | 63,665,633 |
| 2,000 | 66,666,667 | 185,807 | 66,852,474 |
| 2,100 | 70,000,000 | 101,801 | 70,101,801 |
| 2,200 | 73,333,333 | 54,814 | 73,388,147 |
| 2,300 | 76,666,667 | 29,082 | 76,695,749 |
| 2,400 | 80,000,000 | 15,240 | 80,015,240 |
| 2,500 | 83,333,333 | 7,904 | 83,341,237 |

87


# Passive portfolio insurance

Examine initial values.

If index level falls to 500 immediately, portfolio value is 47,088,227.


$$\$47,088,238 = \$50,000,000e^{-.06(1)}$$


| Portfolio values with one year to expiration | | | |
|---|---|---|---|
| Index level | Stock portfolio value | Value of put | Insured portfolio value |
| 500 | 16,666,667 | 30,421,560 | 47,088,227 |
| 600 | 20,000,000 | 27,088,239 | 47,088,239 |
| 700 | 23,333,333 | 23,755,263 | 47,088,596 |
| 800 | 26,666,667 | 20,426,203 | 47,092,869 |
| 900 | 30,000,000 | 17,119,553 | 47,119,553 |
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| 1,200 | 40,000,000 | 8,100,014 | 48,100,014 |
| 1,300 | 43,333,333 | 5,783,980 | 49,117,313 |
| 1,400 | 46,666,667 | 3,948,601 | 50,615,267 |
| 1,500 | 50,000,000 | 2,583,000 | 52,583,000 |
| 1,600 | 53,333,333 | 1,624,612 | 54,957,946 |
| 1,700 | 56,666,667 | 986,331 | 57,652,998 |
| 1,800 | 60,000,000 | 580,381 | 60,580,381 |
| 1,900 | 63,333,333 | 332,300 | 63,665,633 |
| 2,000 | 66,666,667 | 185,807 | 66,852,474 |
| 2,100 | 70,000,000 | 101,801 | 70,101,801 |
| 2,200 | 73,333,333 | 54,814 | 73,388,147 |
| 2,300 | 76,666,667 | 29,082 | 76,695,749 |
| 2,400 | 80,000,000 | 15,240 | 80,015,240 |
| 2,500 | 83,333,333 | 7,904 | 83,341,237 |


88

## Creating protected equity notes

- *Protected equity note* (PEN) is contract that provides guaranteed minimum rate of return on investment plus proportion of price appreciation or total return in underlying stock/index.
  - Also called:
    - Principal protected notes
    - Equity-linked certificates of deposit
    - Bull certificates of deposit

89

## Protected equity notes

- Notation:

  $V \equiv$ principal invested in PEN
  $S \equiv$ index level scaled such that $S = V$
  $g \equiv$ guaranteed investment return on principal
  $k \equiv$ participation rate (e.g., proportion of index gain if market rises)

  - All other notation is as before.

90

## Protected equity notes

- Holding PEN is like holding risk-free bonds plus call option.
  - T-bills provide guaranteed minimum principal.
  - Call provides upside.

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## Protected equity notes

- Guaranteed minimum principal of PEN

$$Ve^{gT}$$

- Number of T-bills to buy today to provide for guarantee is:

$$Ve^{gT}e^{-rT} = Ve^{(g-r)T}$$

92

## Protected equity notes

- Equity share is determined by call with exercise price equal to guaranteed principal, $Ve^{gT}$.

$$c = Se^{-\delta T}N(d_1) - Ve^{gT}e^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln(Se^{-\delta T} / Ve^{(g-r)T}) + .5\sigma^2 T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

93

## Protected equity notes

- BSM formula, however, implies 100% participation rate. Bank states $k$%. Value of PEN is, therefore,

$$\text{PEN} = Ve^{(g-r)T} + kc$$

94

## Protected equity notes

- Illustration:
  - Want invest $100,000 in protected equity note on price appreciation of S&P 500 index for one year.
    - Bank offers guarantee return on principal or 2% plus 30% of price appreciation in S&P 500 index.
    - S&P 500 has:
      - Dividend yield of 1.5%.
      - Volatility rate of 30%.
    - Risk-free interest rate is 6%.
    - Support file: Protected equity note.xlsx

95

## Protected equity notes

- Guaranteed minimum principal of PEN is:

$$Ve^{gT} = 100,000 \times e^{.02(1)} = 102,020.13$$

- Present value of risk-free bonds that protect principal is:

$$Ve^{(g-r)T} = e^{-.06(1)} \times 102,020.13 = 96,078.94$$

96

## Protected equity notes

- Value of call providing 100% participation is:

$$c = 100,000e^{-.015(1)}N(d_1) - 102,020.13e^{-.06(1)}N(d_2) = 7,495.32$$

where

$$d_1 = \frac{\ln\left(100,000e^{-.015(1)}/102,020.13e^{-.06(1)}\right) + .5\left(.16^2\right)1}{.16\sqrt{1}} = .2363,$$

$$d_2 = d_1 - .16\sqrt{1} = .0763$$

97

## Protected equity notes

- Cost of what bank provides is, therefore,

$$\begin{aligned}\text{PEN} &= Ve^{(g-r)T} + kc \\ &= 96,078.94 + .3(7,495.32) = 98,327.54\end{aligned}$$

- Bank's margin is:

$$\text{Margin on PEN} = 100,000 - 98,327.54 = 1,672.46$$

98

## Protected equity notes

- Bank offered 30% of principal. If they offered 30% of total return, set dividend yield rate to 0% and use:

$$c = SN(d_1) - Ve^{(g-r)T} N(d_2)$$

where

$$d_1 = \frac{\ln(S / Ve^{(g-r)T}) + .5\sigma^2 T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

99

## Protected equity notes

- OPTVAL function for valuing protected equity notes.
  - Support file: Protected equity notes.xlsx

E13 =OV_NS_PROTECTED_EQUITY_NOTE($B$3,$B$4,$B$7,$B$6,$B$5,$B$14,$B$10,$B$11,"v")

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Protected equity note** | | | | |
| 2 | *Equity note description* | | | *Analysis* | |
| 3 | Principal ($S$) | 100,000.00 | | Guaranteed floor value ($S\exp(gT)$) | 102,020.13 |
| 4 | Minimum growth rate ($g$) | 2.00% | | Value of risk-free guarantee | 96,078.94 |
| 5 | Years to expiration ($T$) | 1.00 | | Full call option value ($c$) | 8,405.68 |
| 6 | (R)eturn/(P)rice appreciation | R | | Value of promised participation ($kc$) | 2,521.70 |
| 7 | Promised participation rate ($k$) | 30.00% | | Value of protected equity note | 98,600.65 |
| 8 | | | | Embedded margin in dollars | 1,399.35 |
| 9 | *Underlying index/stock:* | | | Embedded margin in percent | 1.399% |
| 10 | Dividend yield ($\delta$) | 1.50% | | Maximum participation rate | 46.65% |
| 11 | Volatility ($\sigma$) | 16.00% | | | |
| 12 | | | | OPTVAL function | |
| 13 | *Market parameters* | | | Value | 98,600.65 |
| 14 | Interest rate ($r$) | 6.00% | | Maximum participation rate | 46.65% |
| 15 | | | | | |

100

## Protected equity notes

- How does bank hedge?
  - Can hedge passively by buying index call options. (cheapest)
  - Can hedge dynamically.

101

## Buffered return enhanced notes

- Protected equity notes evolved into more complicated products.
- Banks now offer high-wealth individuals “special” products with more complicated but “attractive” payoff structures.
  - Buffered enhanced return notes is one example.
- ETF issuers came in and undercut margins.
  - Innovator ETFs

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## Buffered return enhanced notes

- HSBC Buffered Enhanced Notes
  - Note description: HSBC buffered return notes.pdf
  - Support file: HSBC buffered return notes.xlsx

103

## HSBC notes

- HSBC Buffered Enhanced Notes terms
  - Reference asset: S&P 500 index
  - Upside participation rate: 200%
  - Maximum return: 10.74%
  - Buffer amount: 10.00%
  - Downside leverage factor: 1.11111

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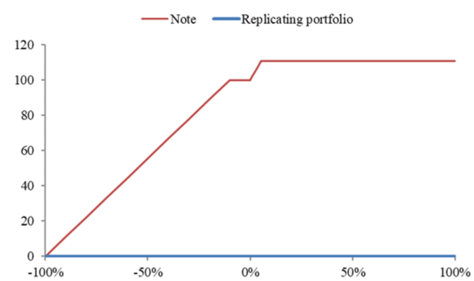
# HSBC notes

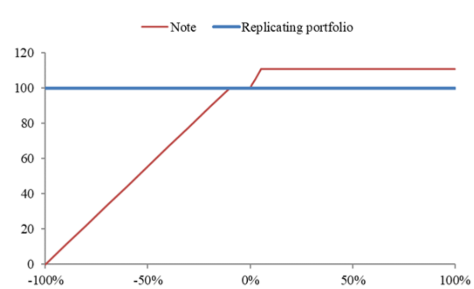
- HSBC Buffered Enhanced Notes payoffs
  - Want to mimic payoffs using exchange-traded options.




105

# HSBC notes

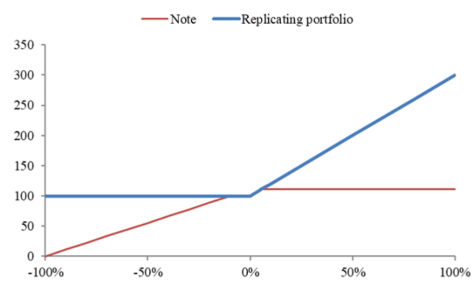
- HSBC Buffered Enhanced Notes payoffs
  - Buy T-bills with face value of 100.




106

# HSBC notes

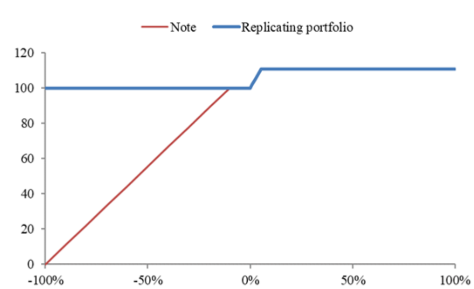
- HSBC Buffered Enhanced Notes payoffs
  - Buy 2 ATM calls.




107

# HSBC notes

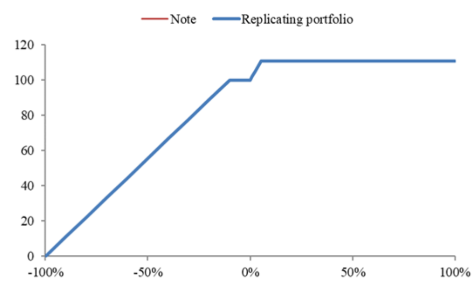
- HSBC Buffered Enhanced Notes payoffs
  - Sell 2 calls with exercise price 5.37.




108

# HSBC notes

- HSBC Buffered Enhanced Notes payoffs
  - Sell 1.11111 puts with exercise price 90.00.




109

# HSBC notes

- HSBC Buffered Enhanced Notes value

**Buffered return enhanced note valuation**

| *Note description* | | *Valuation* | *No. of options* | *Exercise price* | *Total value* |
|---|---|---|---|---|---|
| Principal ($S$) | 100.00 | | | | |
| Cap on index | 5.370% | Buy ATM calls. | 2 | 100.00 | 19.070 |
| Upside leverage factor | 2 | Sell OTM calls. | -2 | 105.37 | -14.610 |
| Maximum return | 10.740% | Sell OTM puts. | -1.1111 | 90.00 | -4.671 |
| Buffered amount | 10.000% | Buy risk-free bonds | | | 97.045 |
| Downside leverage factor | 1.1111 | Value of buffered return enhanced note | | | 96.833 |
| Years to expiration ($T$) | 1.00 | | | | |
| (R)eturn/(P)rice appreciation | P | OptVal function | | | |
| | | Value of buffered return enhanced note | | | 96.833 |
| *Underlying index/stock:* | | | | | |
| Dividend yield ($\delta$) | 1.90% | Margin | | | 3.167 |
| Volatility ($\sigma$) | 23.14% | Margin (%) | | | 3.27% |
| *Market parameters* | | | | | |
| Interest rate ($r$) | 3.00% | | | | |

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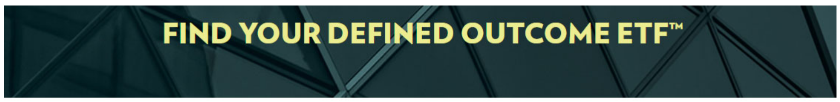
AR15
AR16

# HSBC notes

- ETF industry offers same products at lower cost.
  - http://www.innovatoretfs.com/




All Products | Buffer | Income | Accelerated | Floor | My List | Advanced Filter

As of 11/13/2023 | Download CSV

| TICKER | NAME | SERIES | REFERENCE ASSET | FUND PRICE | FUND RETURN | REF. ASSET RETURN | RETURN DIFF[1] | REF. ASSET RETURN TO CAP[2] | REMAINING CAP | REMAINING BUFFER | DOWNSIDE BEFORE BUFFER | REMAINING OUTCOME PERIOD |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| TJUL | Equity Defined Protection ETF | July 2025 | SPY | $24.73 | -0.54% | -2.36% | 1.82% | 19.44% | 16.96% | 100.29% | 0.00% | 595 days |
| UNOV | U.S. Equity Ultra Buffer ETF | November | SPY | $30.51 | 2.82% | 5.26% | -2.44% | 9.68% | 12.24% | 30.61% | -7.63% | 353 days |
| INOV | Intl Developed Power Buffer ETF | November | EFA | $25.54 | 2.36% | 3.54% | -1.18% | 16.01% | 17.31% | 15.57% | -2.34% | 353 days |
| BNOV | U.S. Equity Buffer ETF | November | SPY | $33.22 | 3.86% | 5.25% | -1.39% | 14.54% | 16.04% | 9.79% | -3.75% | 353 days |

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# HSBC notes

- ETF industry offers same products at lower cost.
  - http://www.innovatoretfs.com/

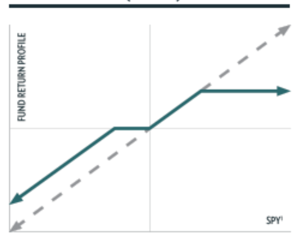
**DETAILS**

| | |
|---|---|
| Ticker | BFEB |
| Listing date | February 3, 2020 |
| Exposure | SPY |
| Starting cap | 19.47% |
| Starting buffer | 9% |
| Outcome period | 2/1/2023-1/31/2024 |
| Rebalance frequency | Annual |
| Expense ratio | $\boxed{0.79\%}$ |
| Net assets | $98.42M |
| Exchange | Cboe BZX |
| Series | February |

**KEY POINTS**

- Defined downside buffer levels
- Exposure to S&P 500 via the SPY's upside performance (to a cap)
- Cost effective, flexible, liquid, and transparent
- Tax-efficient†
- No credit risk†
- Rebalances annually and can be held indefinitely

**PAYOFF PROFILE (1 YEAR)**




*For information purposes only. Does not represent actual fund performance. Intended to illustrate the return profile the investment objective seeks to achieve relative to SPY. Illustration does not account for fund fees and expenses.*

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## Slide 111

**AR15** Update screenshots of the follwing slides
Reams, Allen J, 11/13/2023

**AR16** Updated. Make sure these products are acceptable
Reams, Allen J, 11/13/2023

# HSBC notes

- ETF industry offers same products at lower cost.
  - http://www.innovatoretfs.com/

**DETAILS**

| | |
|---|---|
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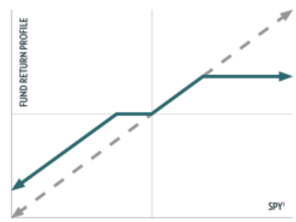


- SPY
- Innovator U.S. Equity Buffer ETF™

*For information purposes only. Does not represent actual fund performance. Intended to illustrate the return profile the investment objective seeks to achieve relative to SPY. Illustration does not account for fund fees and expenses.*

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# Lesson summary

- Stock index futures were introduced in U.S. in 1982; stock index options in 1983.
- Construction of stock indexes
  - Value-weighted indexes
  - Price-weighted indexes
- Cash dividends
  - Use discrete cash flows in valuation

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## Lesson summary

- Stock index arbitrage
  - Program trading
- Simple OLS regression of portfolio returns on futures returns provides estimate of minimum-risk hedge ratio.
- Multiple OLS regression can be used to estimate hedge ratios for portfolios with multiple sources of risk.

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## Lesson summary

- Steps in setting multiple risk portfolio hedge:
  - Identify portfolio risk exposures and find futures to proxy for each.
  - Collect historical return series.
  - Estimate OLS regression.
    - Use coefficient estimates to hedge selected exposures.

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## Lesson summary

- Careful data analysis is necessary to properly estimate hedge ratios. Examined effects of:
  - Data frequency
  - Missing variables bias
    - Failing to include relevant explanatory variable is much more serious than including irrelevant explanatory variable.

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## Lesson summary

- Dominant markets are S&P 500 option markets, same as futures.
- Applications of stock index options
  - Insure stock portfolios.
  - Create protected equity notes.
  - Create buffered return enhanced notes.
    - From private OTC notes to exchange-traded products

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