No-arbitrage price relations

Options

No-arbitrage relations

Asset market	Asset option market
Futures market	Futures option market

No-arbitrage relations

Asset market

Asset option market

Four markets operate simultaneously.

Futures market

Futures option market

No-arbitrage relations

Stock index market (NYSE, Nasdaq)	Asset market	Asset option market	Stock index option market (Cboe)
Stock index futures market (CME)	Futures market	Futures option market	Stock index futures option market (CME)

No-arbitrage relations

FX market (OTC)	Asset market	Asset option market	FX option market (PHLX)
FX futures market (CME)	Futures market	Futures option market	FX futures option market (CME)

No-arbitrage relations

Asset market

SEC

Asset option market

Only in US because of two regulatory bodies, SEC and CFTC.

Futures market

CFTC

Futures option market

No-arbitrage relations

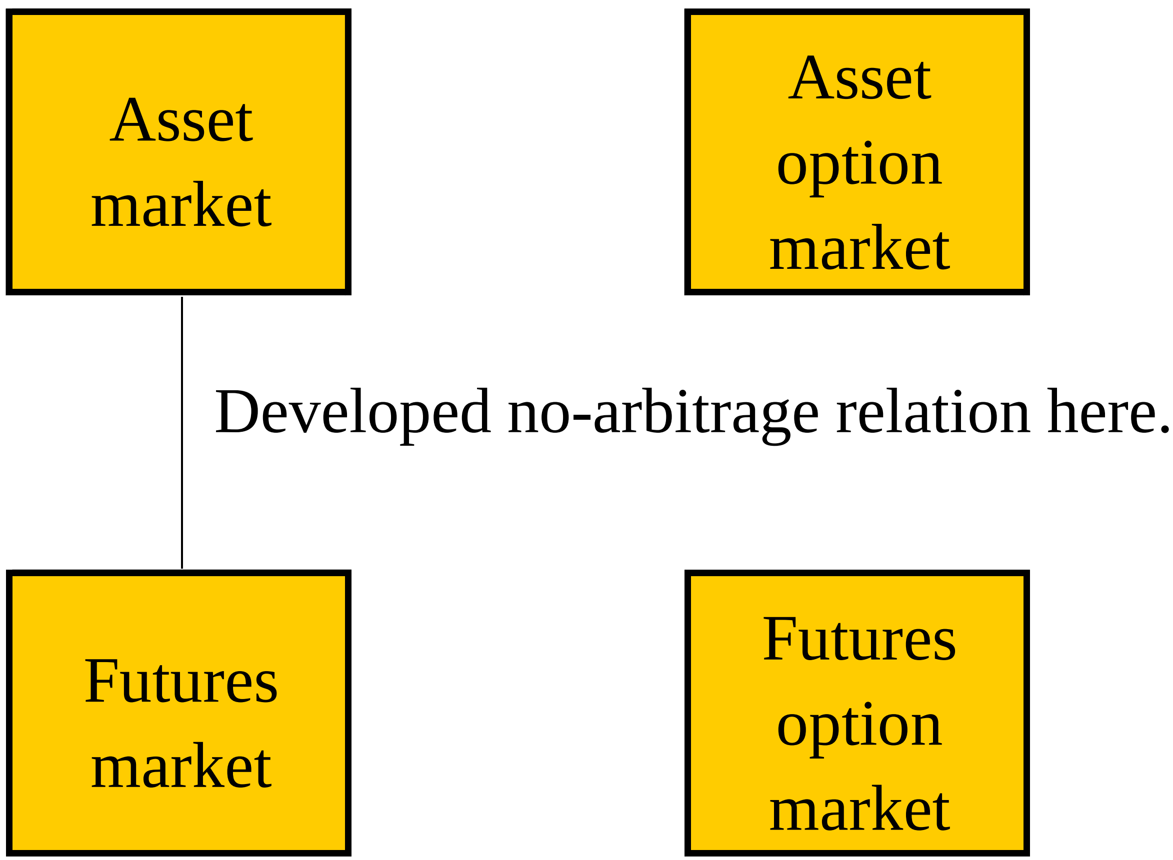
Asset market

Asset option market

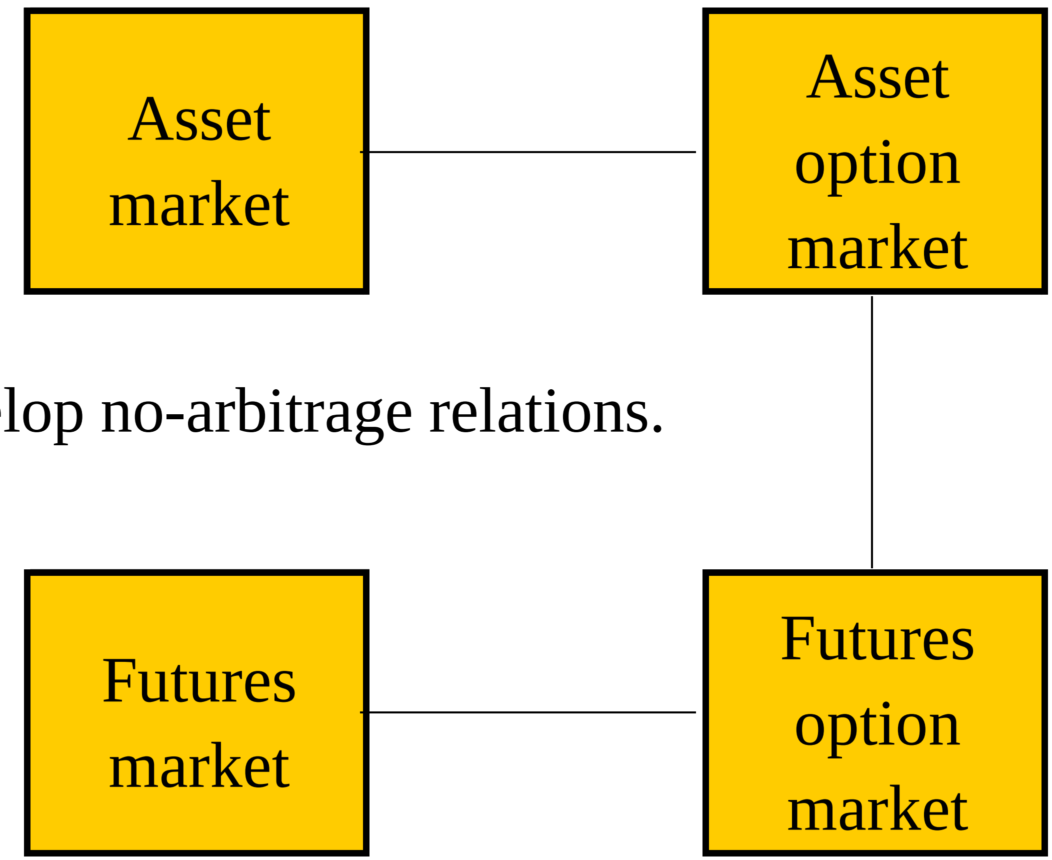
In all other major financial centers, one regulatory authority.

Futures market

No-arbitrage relations



No-arbitrage relations



Will develop no-arbitrage relations.

No-arbitrage relations

- Purpose:
  - Develop no-arbitrage price relations for options.
  - Key results are:
    - put-call parity relations
    - inter-market price relations

No-arbitrage relations

- Assumptions:
  - Individuals are rational.
    - Prefer more wealth to less wealth.
    - No costless arbitrage opportunities.
  - Markets are frictionless.

No-arbitrage relations

- Assumptions:
  - Futures, option, and futures option contracts expire at same time T.
  - Individuals can borrow and lend at constant and continuous rate r.
  - Non-interest carry costs/benefits rate i of asset is constant and continuous.

No-arbitrage relations

- Terminology:
  - *Option* provides right to buy or sell underlying asset at specified price within certain period.
    - *Call option* is right to buy.
    - *Put option* is right to sell.

No-arbitrage relations

- Terminology:
  - Specified price at which asset is bought or sold is called *exercise price* or *strike price*.
  - Two styles of options are traded:
    - *European-style*: exercised only at expiration
    - *American-style*: exercised at any time prior to expiration
  - Focus only on European-style options.
    - American-style results are tractable but more complicated.

No-arbitrage relations

- Terminology:
  - if $S>X$,
    - call is *in-the-money*
    - put is *out-of-the-money*
  - if $S<X$,
    - call is *out-of-the-money*
    - put is *in-the-money*
  - if $S=X$,
    - call and put are *at-the-money*

No-arbitrage relations

- Notation:

$$S(\tilde{S}_T) = \text{ current (random terminal) asset price}$$

$$F(\tilde{F}_T) = \text{ current (random terminal) futures price}$$

$$c = \text{ European-style call price}$$

$$p = \text{ European-style put price}$$

$$X = \text{ exercise price of price}$$

$$T = \text{ time to expiration of option}$$

Put-call parity

- Put-call parity for options on assets with continuous income rate *i*.

$$c - p = Se^{-iT} - Xe^{-rT}$$

- Put-call parity for options on assets with discrete income payment *I* at time *t*.

$$c - p = S - I_t e^{-rt} - Xe^{-rT}$$

Put-call parity

- Asset options
  - Form arbitrage portfolio using asset and options.

Action	Initial value	Terminal value $\tilde{S}_T \le X$	Terminal value $\tilde{S}_T > X$
Buy asset	$-Se^{-iT}$	$\tilde{S}_T$	$\tilde{S}_T$
Buy European put	$-p$	$X-\tilde{S}_T$	0
Sell European call	c	0	$-(\tilde{S}_T - X)$
Sell T-bills (borrow)	Xe^{-rT}	$-X$	$-X$
Net value	$-Se^{-iT}-p+c+Xe^{-rT}$	0	0

Put-call parity

- Asset options
  - Buy e^{-iT} units of asset.

		Terminal value	
Action	Initial value	$\tilde{S}_T \le X$	$\tilde{S}_T > X$
Buy asset	$-Se^{-iT}$	$\tilde{S}_T$	$\tilde{S}_T$

Put-call parity

- Asset options
  - Buy one put option.

		Terminal value	
Action	Initial value	$\tilde{S}_T \le X$	$\tilde{S}_T > X$
Buy asset	$-Se^{-iT}$	$\tilde{S}_T$	$\tilde{S}_T$
Buy European put	$-p$	$X-\tilde{S}_T$	0

Put-call parity

- Asset options
  - Sell one call option.

Action	Initial value	Terminal value: $\tilde{S}_T \leq X$	Terminal value: $\tilde{S}_T > X$
Buy asset	$-Se^{-iT}$	$\tilde{S}_T$	$\tilde{S}_T$
Buy European put	$-p$	$X - \tilde{S}_T$	0
Sell European call	c	0	$-(\tilde{S}_T - X)$

Put-call parity

- Asset options
  - Borrow e^{-rT} units of risk-free discount bonds with face value X.

		Terminal value	
Action	Initial value	$\tilde{S}_T \le X$	$\tilde{S}_T > X$
Buy asset	$-Se^{-iT}$	$\tilde{S}_T$	$\tilde{S}_T$
Buy European put	$-p$	$X-\tilde{S}_T$	0
Sell European call	c	0	$-(\tilde{S}_T - X)$
Sell T-bills (borrow)	Xe^{-rT}	$-X$	$-X$

Put-call parity

- Asset options
  - Compute net terminal values.

Action	Initial value	Terminal value $\tilde{S}_T \le X$	Terminal value $\tilde{S}_T > X$
Buy asset	$-Se^{-iT}$	$\tilde{S}_T$	$\tilde{S}_T$
Buy European put	$-p$	$X-\tilde{S}_T$	0
Sell European call	c	0	$-(\tilde{S}_T - X)$
Sell T-bills (borrow)	Xe^{-rT}	$-X$	$-X$
Net value	$-Se^{-iT}-p+c+Xe^{-rT}$	0	0

Put-call parity

- Asset options
  - Compute net terminal values.

		Terminal value	
Action	Initial value	$\tilde{S}_T \leq X$	$\tilde{S}_T > X$
Buy asset	$-Se^{-iT}$	$\tilde{S}_T$	$\tilde{S}_T$
Buy European put	$-p$	$X - \tilde{S}_T$	0
Sell European call	c	0	$-(\tilde{S}_T - X)$
Sell T-bills (borrow)	Xe^{-rT}	$-X$	$-X$
Net value	$-Se^{-iT} - p + c + Xe^{-rT}$	0	0

Since portfolio is guaranteed to have 0 terminal value, initial
Value must be 0; other costless arbitrage profits are possible.

Put-call parity

- Asset options
  - Compute net terminal values.

Action	Initial value	Terminal value $\tilde{S}_T \le X$	Terminal value $\tilde{S}_T > X$
Buy asset	$-Se^{-iT}$	$\tilde{S}_T$	$\tilde{S}_T$
Buy European put	$-p$	$X-\tilde{S}_T$	0
Sell European call	c	0	$-(\tilde{S}_T - X)$
Sell T-bills (borrow)	Xe^{-rT}	$-X$	$-X$
Net value	$-Se^{-iT}-p+c+Xe^{-rT}$	0	0

Setting initial value to 0, market is in equilibrium where

$$c - p = Se^{-iT} - Xe^{-rT}$$

Put-call parity

- Illustration:
  - Consider call and put with:
    - exercise price of 70
    - time to expiration of 90 days
    - prices of 5.00 and 4.50
  - Assume:
    - asset price is 70
    - income rate is 1%
    - interest rate is 3%

Put-call parity

- <u>Illustration</u>:

$$5.00 - 4.50 = .50 > 70e^{-.01(90/365)} - 70e^{-.03(90/365)} = .34$$

0.16 costless arbitrage opportunity available.

Put-call parity

- Illustration:
  - Implies any or all of:
    - Call is overpriced.
    - Put is underpriced.
    - Asset is underpriced.
  - Arbitrage portfolio must account for all possibilities (risks).

Put-call parity

- Illustration:

Action	Initial value	Terminal value $\tilde{S}_T \le 70$	Terminal value $\tilde{S}_T > 70$
Buy asset	$-70e^{-.01(90/365)} = -69.83$	$\tilde{S}_T$	$\tilde{S}_T$
Buy European put	-4.50	$70-\tilde{S}_T$	0
Sell European call	5.00	0	$-(\tilde{S}_T-70)$
Sell T-bills (borrow)	$70e^{-.03(90/365)}$	-70	-70
Net value	$.16$	0	0

Put-call parity

- Illustration: Put-call parity.xlsx

	A	B	C	D	E	F	G
1	EUROPEAN-STYLE OPTIONS ON ASSETS WITH CONTINUOUS INCOME RATE						
2	ASSET			COSTLESS ARBITRAGE PORTFOLIO			
3	Price (S)	70.00				Terminal value	
4	Income rate (i)	1.00%		Action	Initial value	ST<=70	ST > 70
5				Buy asset	-69.828	ST	ST
6	OPTIONS			Buy put	-4.500	70-ST	0
7	Exercise price	70		Sell call	5.000	0	-(ST-70)
8	Days to expiration	90		Sell riskless bonds	69.484	-70	-70
9	Time to expiration (T)	0.24658		Net value	0.156	0	0
10	Call price (c)	5.000					
11	Put price (p)	4.500					
12							
13	INTEREST RATE (r)	3.00%					
14							
15	Sexp(-iT)	69.828					
16	Xexp(-rT)	69.484					
17							
18	PUT-CALL PARITY						
19	Deviation	0.156					
20	Costless arbitrage profit	0.156					

Put-call parity

- Illustration:
  - Consider European-style call and put with:
    - exercise price of 70
    - time to expiration of 90 days
    - prices of 5.00 and 4.50
  - Assume:
    - asset price is 70
    - income payment is 2.00 in 20 days
    - interest rate is 3%

Put-call parity

- Illustration: Put-call parity.xlsx

	A	B	C	D	E	F	G
1	EUROPEAN-STYLE OPTIONS ON ASSETS WITH DISCRETE INCOME PAYMENT						
2	ASSET			COSTLESS ARBITRAGE PORTFOLIO			
3	Price (S)	70.00				Terminal value	
4	Income payment (I)	2.00		Action	Initial value	ST<=70	ST > 70
5	Days to income payment	20		Buy asset	-70.000	ST	ST
6	Time to income payment (t)	0.0548		Receive dividend	1.997		
7				Buy put	-4.500	70-ST	0
8	OPTIONS			Sell call	5.000	0	-(ST-70)
9	Exercise price	70		Sell riskless bonds	69.484	-70	-70
10	Days to expiration	90		Net value	1.981	0	0
11	Years to expiration (T)	0.2466					
12	Call price (c)	5.000					
13	Put price (p)	4.500					
14							
15	INTEREST RATE (r)	3.00%					
16							
17	PUT-CALL PARITY						
18	Deviation	1.981					
19	Costless arbitrage profit	1.981					
20							

Put-call parity

- European-style asset options

$$c - p = Se^{-iT} - Xe^{-rT}$$

- Prepaid cost of carry relation

$$Fe^{-rT} = Se^{-iT}$$

- European-style futures options

$$c - p = e^{-rT}\left(F - X\right)$$

Put-call parity

- Futures options
  - Consider portfolio:

Action	Initial value	Terminal value: $\tilde{F}_T \le X$	Terminal value: $\tilde{F}_T > X$
Buy futures	0	$\tilde{F}_T - F$	$\tilde{F}_T - F$
Buy European put	$-p$	$X - \tilde{F}_T$	0
Sell European call	c	0	$-(\tilde{F}_T - X)$
Sell T-bills (borrow)	$-\left(F - X\right)e^{-rT}$	$F - X$	$F - X$
Net value	$-\left(F - X\right)e^{-rT} - p + c$	0	0

Put-call parity

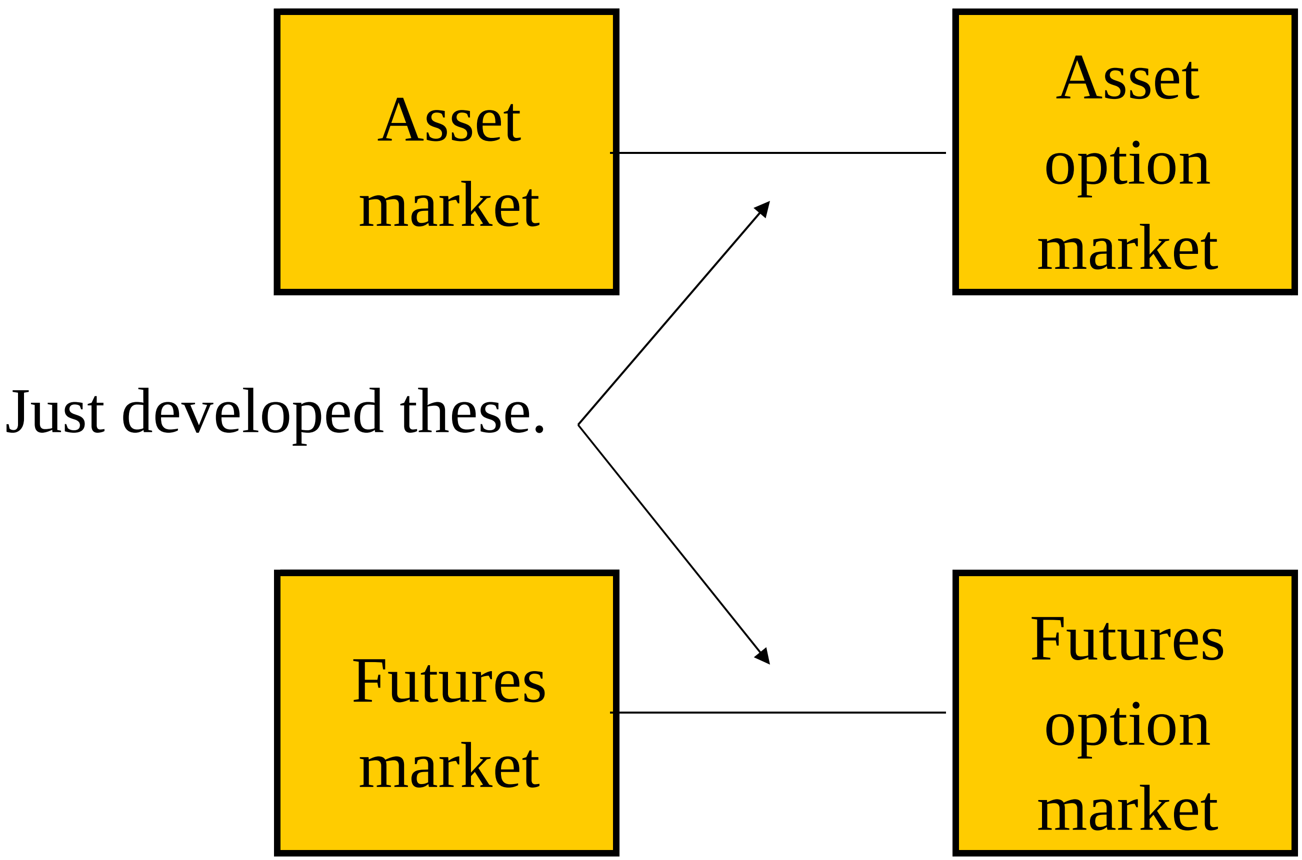
- Illustration:
  - Consider European-style call and put on futures with:
    - exercise price of 70
    - time to expiration of 90 days
    - prices of 5.00 and 4.50
  - Assume:
    - futures price is 70
    - interest rate is 3%

Put-call parity

- European-style futures options
  - Consider portfolio:

	A	B	C	D	E	F	G
1				EUROPEAN-STYLE OPTIONS ON FUTURES			
2	FUTURES			COSTLESS ARBITRAGE PORTFOLIO			
3	Price (F)	70.00				Terminal value	
4				Action	Initial value	FT<=70	FT > 70
5	OPTIONS			Buy futures		FT-70	FT-70
6	Exercise price	70		Buy put	-4.500	70-FT	0
7	Days to expiration	90		Sell call	5.000	0	-(FT-70)
8	Time to expiration (T)	0.24658		Buy riskless bonds	0.000	0.000	0.000
9	Call price (c)	5.000		Net value	0.500	0	0
10	Put price (p)	4.500					
11							
12	INTEREST RATE (r)	3.00%					
13							
14	PUT-CALL PARITY						
15	Deviation	0.500					
16	Costless arbitrage profit	0.500					

No-arbitrage relations



Inter-market relations

- Asset and futures options
  - Since futures, options and futures options expire at same time, it follows:

$$c(F) = c(S)$$

$$p(F) = p(S)$$

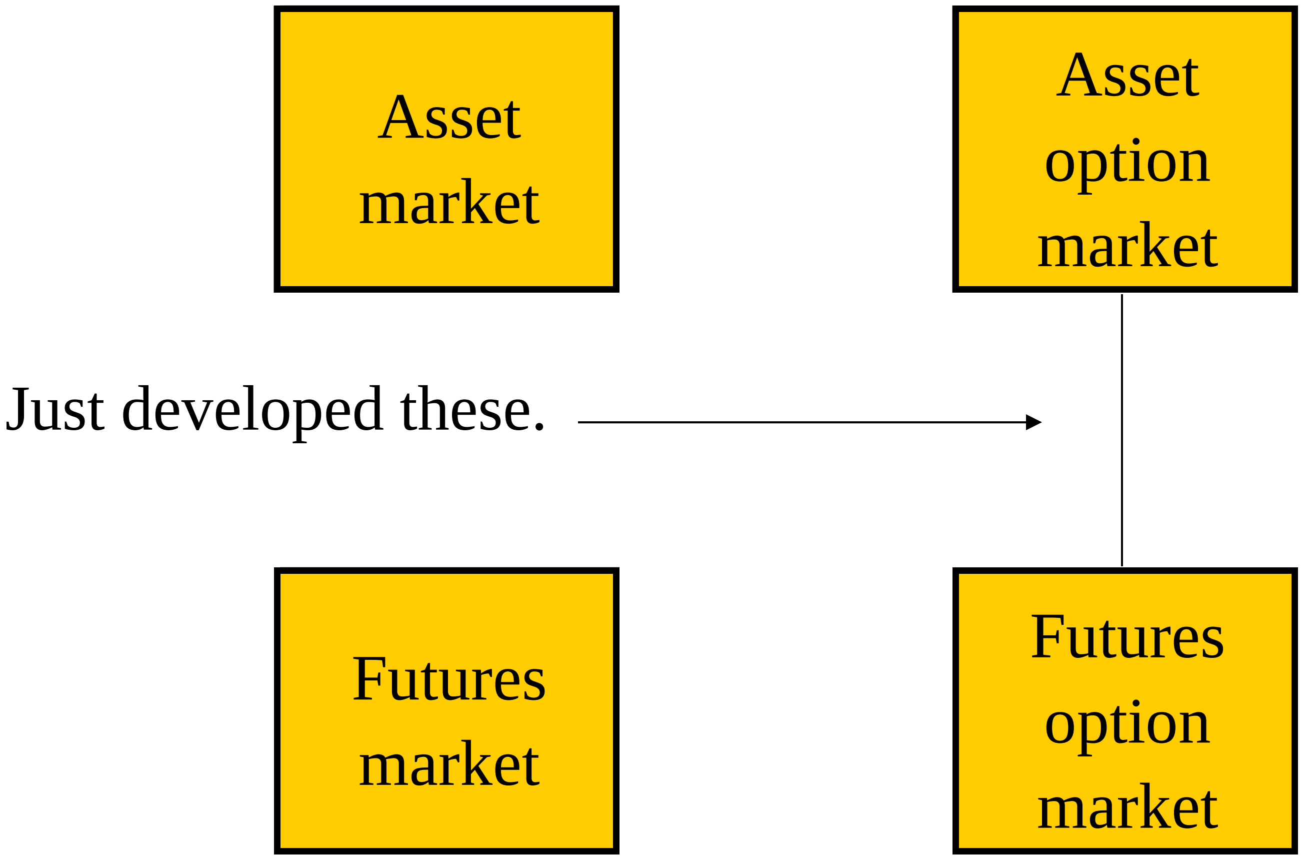
Inter-market relations

- Calls

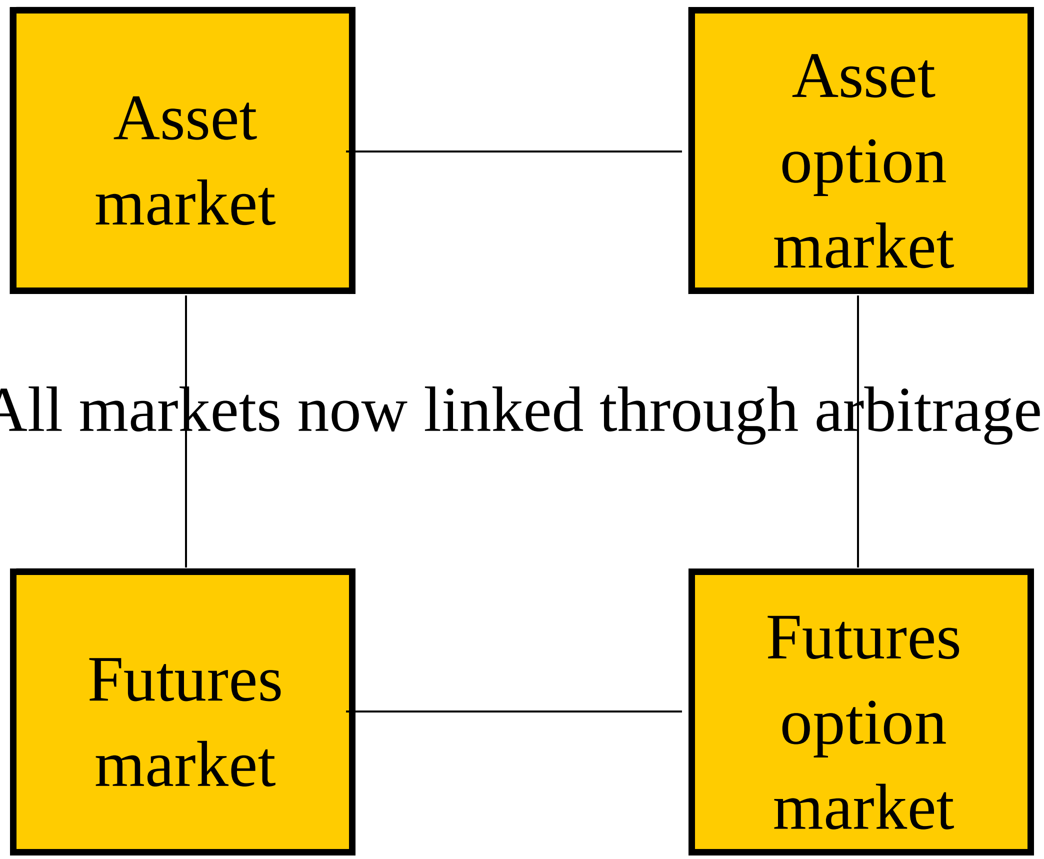
$$c(F) = c(S)$$

Action	Initial value	Terminal value $\tilde{S}_T \leq X$	Terminal value $\tilde{S}_T > X$
Sell call on asset	$c(S)$	0	$-(\tilde{S}_T - X)$
Buy call on futures	$-c(F)$	0	$\tilde{S}_T - X$
Net value	$c(S) - c(F)$	0	0

No-arbitrage relations



No-arbitrage relations



All markets now linked through arbitrage.

Lesson summary

- No-arbitrage pricing relations exist linking all derivatives to underlying asset price.
- Most commonly applied in practice are:
  - Put-call parity
  - Inter-market price relations