# ON THE VALUATION OF AMERICAN PUT OPTIONS ON DIVIDEND-PAYING STOCKS

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## I. INTRODUCTION

The option pricing literature has developed enormously since Black–Scholes [2] (Merton [8]) derived valuation equations for the European call and put options written on zero (constant proportional) dividend yield stocks. Models now exist for pricing European and American options on a variety of underlying commodities ranging from financial assets such as common stocks and bonds to traditional agricultural futures contracts.[1]

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This research was supported by the Futures and Options Research Center, Duke University, and the SSHRC.

Advances in Futures and Options Research, Vol. 3, pages 1–13.

ISBN: 0-89232-926-2

Conspicuously absent from this literature is the valuation of American options written on commodities that have discrete cash payments during the life of the option. In particular, while an analytic valuation equation exists for the American call option written on a stock with known discrete dividends paid during the option's life (see Roll [10] and Whaley [12]), the valuation of the American put option on a stock with discrete dividends remains largely unaddressed. Usually the valuation of such put options involves implementing finite difference methods, and these methods are computationally expensive and impractical for real time pricing applications. Blomeyer [3] devised a fast algorithm that interpolates between known option values that surround the true unknown price. Unfortunately, however, his technique can lead to serious mispricing errors for typical parameter ranges. The purpose of this paper is to provide a fast and accurate approximation for the value of an American put option on a stock with known discrete dividend payments.

The outline of the paper is as follows. In Section II, the economics of the American put option valuation problem are discussed. First, the assumptions and definitions used in the analysis are presented. The problem is then formulated and the approximation method is presented and discussed. In Section III, the approximation method is applied and the results are compared with "true" put option values. For the stock option parameter ranges considered, the maximum pricing error is less than 1% and falls within the bid–ask spread. The paper concludes with a brief summary (Section IV).

## II. THEORETICAL APPROXIMATION

### A. Assumptions and Definitions

The approximation method relies on all of the standard Black–Scholes [2] assumptions, except that, in place of the no dividend assumption, it is assumed that the underlying stock pays a *single known* cash dividend during the option's life. Extensions to multiple known dividends are straightforward. The approximation itself, however, employs the valuation equations for European and American put options on stocks with no dividends. From Black–Scholes [2], the European put option formula is

$$p(S,T) = Xe^{-rT}N_1(-d_2) - SN_1(-d_1), \tag{1}$$

where

$$d_1 = \frac{\ln(S/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{1a}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{1b}$$

In Equation (1), S is the current stock price, X is the exercise price of the option, r is the riskless rate of interest, $\sigma$ is the standard deviation of the instantaneous rates of return on the underlying stock, and T is the time to expiration of the option. $N_1(d)$ is the cumulative univariate normal density function with upper integral limit d. From MacMillan [7], the American put option valuation equation[2] is

$$P(S, T) = \begin{cases} p(S,T) + A_1(S/S^*)^{q_1} & \text{where } S > S^*, \\ X - S & \text{where } S \leqslant S^*, \end{cases} \tag{2}$$

where

$$A_1 = -\frac{S^*\{1 - N[-d_1(S^*)]\}}{q_1}, \tag{2a}$$

$$q_1 = \frac{1 - n - \sqrt{(n-1)^2 + 4k}}{2}, \tag{2b}$$

$$n = \frac{2r}{\sigma^2}, \tag{2c}$$

$$k = \frac{2r}{\sigma^2(1 - e^{-rT})}, \tag{2d}$$

and $S^*$ is the critical asset below above which the American put should be exercised immediately and is the solution to

$$X - S^* = p(S^*, T; X) - \{1 - N_1[-d_1(S^*)]\}S^*/q_1. \tag{2e}$$

In the approximation method for the American put option on a dividend-paying stock, the European and American put option valuation Equations (1) and (2) are used repeatedly for various stock prices and times to expiration. In addition, it is necessary to compute critical stock prices below which the American put will be exercised immediately. Henceforth, the notation $S^*(T)$ denotes the critical stock price below which the American put option written on a non-dividend-paying stock will be exercised immediately.

## B. Put Option Pricing Problem

Before explaining the approximation method, the nature of the pricing problem for the American put option on a dividend-paying stock is discussed. To begin, dividends are ignored. In the absence of dividend payments on the underlying stock, the American put option may be exercised early because interest income can be earned on the exercisable proceeds of the option as soon as the option is exercised. Deferring exercise implicitly means interest income is being foregone. When the stock pays a dividend, however, the American put option holder is in a dilemma. If he continues to hold the put, he foregoes the interest; but, if he exercises immediately, he will not profit from the discrete upward jump in the exercisable proceeds of the put when the stock goes ex-dividend.

The tradeoff between the interest income and the dividend can be expressed more formally. Assume that the stock pays a dividend D at time $t_D$ during the option's life. If the stock is zero today and the put option is exercised, the interest income that would be earned between now and the ex-dividend instant is $X(e^{rt_D} - 1)$. Now, if the dividend amount D at $t_D$ is larger than the interest income, that is, if

$$D > X(e^{rt_D} - 1), \tag{3}$$

early exercise during the interval 0 to $t_D$ is not rational and, if the American put is exercised at all, it will be during the time interval $t_D$ to T.

Condition (3) can be used to gather further insight about the put option pricing problem. To do so, define $t_N$ as a point in time before which it may be optimal to exercise the put early prior to ex-dividend, but after which it will not be optimal to exercise until after the dividend has been paid. This point in time is defined by the solution to

$$D = X[e^{r(t_D - t_N)} - 1], \tag{4}$$

and is

$$t_N = t_D - \frac{\ln(1 + D/X)}{r}. \tag{5}$$

Now, if $t_N$ is negative, condition (3) holds and the American put will not be exercised during the interval 0 to $t_D$ and may be exercised during the interval $t_D$ to T. On the other hand, if $t_N$ is positive, there exist three distinct time intervals during the option's life with different early exercise implications: during the interval 0 to $t_N$, there is a nonzero probability of early exercise; during the interval $t_N$ to $t_D$, the probability of early exercise is zero; and during the interval $t_D$ to T, there is a nonzero probability of early exercise.

## C. Approximate Solution

The approximate solution to the put option pricing problem described in Section II,B has two parts—first where early exercise before the dividend is not possible and second where early exercise before the dividend is possible. In both cases, the approach to finding the solution is the same. Known option prices to which the true put option value $P(S, T; D, t_D)$ converges are identified and then weighted by the "probabilities" of their occurrence. The weighted average approximates the true option value.

The first case to be examined is where early exercise is precluded during the interval 0 to $t_D$, but is possible during the interval $t_D$ to T (i.e., the case where $t_N \leqslant 0$). This occurs typically when the dividend paid during the option's life is large relative to the exercise price of the option. An upper bound for the put option $P(S, T; D, t_D)$ is the value of an American put written on a non-dividend-paying stock [i.e., Equation (2)], where the stock price net of the present value of the escrowed dividend, $S' = S - De^{-rt_D}$, replaces the stock price parameter, $P(S', T)$. This option necessarily overstates the true option value since it prices an early exercise premium over the entire option's life, when the early exercise premium between 0 and $t_D$ is known to be equal to zero. But, as the stock price becomes large, the early exercise premium becomes small and the true value of the put converges to $P(S', T)$.

A lower price bound for $P(S, T; D, t_D)$ is found by subtracting the early exercise premium from 0 to $t_D$ from the upper bound, that is,

$$P(S', T) - \varepsilon_P(S', t_D), \tag{6}$$

where the early exercise premium is defined by

$$\varepsilon_P(S', t_D) = P(S', t_D) - p(S', t_D). \tag{6a}$$

If the early exercise premium of an option decreased linearly as the option's life erodes (holding other factors constant), the lower price bound, $P(S', T) - \varepsilon_P(S', t_D)$, would, in fact, be the true price of the put. But, because the early exercise premium of an option decreases at an increasing rate,[3] the early exercise premium (6a) is larger than the early exercise premium imbedded in the American put and therefore the option value (6) provides an understatement of the true value of the put. The true option value converges to (6) as the stock price falls and the probability of early exercise at $t_D$ grows large.

The upper and lower bounds of the true price are now weighted to provide an approximation for the true value. The approximation formula is

$$P(S, T; D, t_D) = w_1 P(S', T) + w_2 [P(S', T) - \varepsilon_P(S', t_D)], \tag{7}$$

where $w_1$ and $w_2$ sum to one.[4] The weight $w_2$ is the probability that the put will be exercised immediately after the dividend is paid.[5] To compute this probability, Equation (2) is first used to evaluate the price of an American put with time to expiration $T - t_D$. A by-product of the valuation is the computation of the critical stock price $S^*(T - t_D)$ below which the American put option holder will exercise at $t_D$. Note that the critical stock price is not a function of the stock price itself. With the critical stock price in hand, the probability of early exercise at $t_D$ is then computed as

$$w_2 = N_1(-b) \tag{7a}$$

where

$$b = \frac{\ln[S'/S^*(T-t_D)] + (r - .5\sigma^2)t_D}{\sigma\sqrt{t_D}}. \tag{7b}$$

The weight,

$$w_1 = N_1(b), \tag{7c}$$

is the complement of (7a) or the probability that the put will not be exercised at time $t_D$.

It is instructive to note that the approximation (7) converges to its proper limits as the option goes deep in-the-money and out-of-the-money. As the put goes deep in-the-money (i.e., as the stock price falls), $w_1$ goes to zero, $w_2$ goes to one, and the value of (6) converges to the European put option values, $p(S', t_D)$.[6] This stands to reason. If $t_N \leqslant 0$, early exercise will not occur prior to ex-dividend. As the stock price moves lower and lower, the likelihood of exercising the put just after the dividend is paid approaches one. If exercise at $t_D$ is certain, the American put option will have the same value as a European put expiring at $t_D$ just after the dividend is paid, that is, $p(S', t_D)$. On the other hand, as the put goes deep out-of-the-money (i.e., the stock price rises), $w_1$ goes to one, $w_2$ goes to zero, and the American put price $P(S', T)$ approaches the European put price $p(S', T)$. That is, the deeper out-of-the-money the put option is, the lower the early exercise premium is, until eventually the American put is priced as if it were European.

The case where $t_N$ is positive is slightly more complex because the premium arising from the prospect of early exercise between 0 and $t_N$ must be recognized. If early exercise is certain to occur during the interval 0 to $t_N$, pricing the American put would be the straightforward problem of pricing an American put option on a non-dividend-paying stock. The price of such an option can be approximated using Equation (2), that is, $P(S, t_N)$. But, in general, early exercise between 0 and $t_N$ is not certain so this additional put must be combined probabilistically with the other two in Equation (7). When $t_N > 0$, the approximation for the American put is

$$P(S, T; D, t_D) = w_1 P(S', T) + w_2[P(S', T) - \varepsilon_p(S', t_D)] + w_3 P(S, t_N), \tag{8}$$

where $w_1 + w_2 + w_3 = 1$.[7] The weight $w_3$ is the probability that the put will be exercised in the interval 0 to $t_N$, that is,

$$w_3 = N_1(-a) \tag{8a}$$

where

$$a = \frac{\ln[S/S^*(T - t_N)] + (r - .5\sigma^2)t_N}{\sigma\sqrt{t_N}}. \tag{8b}$$

Note that this probability is only an approximation in the sense that it is the probability that a put option written on a non-dividend-paying stock will be exercised at time $t_N$[8] [as reflected by the use of the critical stock price $S^*(T - t_N)$]. The weights $w_1$ and $w_2$ remain as above except that they are now joint probabilities. In order for the put life's to be extended beyond $t_N$, the weights must reflect the probability that the put is not exercised at $t_N$, that is, $N(a)$. The weights are now

$$w_1 = N_2(a, b; \sqrt{t_N/t_D}) \tag{8c}$$

and

$$w_2 = N_2(a, -b; -\sqrt{t_N/t_D}), \tag{8d}$$

where $N_2(a, b; \rho)$ is the cumulative bivariate normal density function with upper integral limits a and b and correlation coefficient $\rho$.[9] Note that the weights $w_1 + w_2$ sum to the probability of no early exercise at $t_N$, $N_1(a)$.

## III. SIMULATION RESULTS

Before proceeding with a discussion of the accuracy of the approximation based on some simulation results, a simple example is constructed to illustrate the technique. Suppose that an American put option has an exercise price of 40 and a time to expiration of 0.3288 years. Assume that the current stock price is 40, that the stock pays a dividend of 0.50 exactly one-half way through the option's life, and that the standard deviation of the instantaneous rates of return of the stock net of the present values of the escrowed dividend is 20% annually. Assume also that the stock riskless rate of interest is 10% annually.

The first step is to compute the value of $t_N$ using Equation (5). In this example, $t_N = 0.0402$. Since $t_N$ is positive, Approximation Equation (8) is used. In (8), the critical stock price below which the put will be exercised at $t_N$, $S^*(T - t_N)$, is 35.96. The probability that the stock price will move from its current level of 40 to a value below 35.96 at $t_N$, $w_3$ is 0.0031, and the value of the American put expiring at $t_N$, $P(S, t_N)$, is 0.5724. If the put is not exercised at $t_N$, it may be exercised at $t_D$. The probability that the put will not be exercised at $t_N$ and will be exercised at $t_D$, $w_2$, is 0.1271, and the value, $P(S', T) - \varepsilon_P(S', t_D)$ is $1.5735 - (1.2724 - 1.1966) = 1.4977$. Finally, the probability that the put is not exercised at $t_N$ and is not exercised at

$t_D, w_1$, is 0.8698, and the value of the option, $P(S',T)$, is 1.5735. Substituting the numerical values into Equation (8),

$$P(40,.3288;.50,.1644) = .8698(1.5735)2.1271(1.4977)2.0031(.5724) = 1.5607.$$

This put option value is reported in row 10 of Table 1.

Tables 1 and 2 contain the results of some simulations using the proposed nonlinear interpolation method. For comparison purposes, the values of the

*Table 1.* Comparison of American Put Option Approximation Values[a] ($S = 40, r = .10, D = 0.50, t = T/2$)

| *No.* | *X* | *T* | *σ* | *APPX1*[b] | *APPX2*[c] | *True Value*[d] | *APPX3*[e] |
|---|---|---|---|---|---|---|---|
| 1 | 35 | 0.1644 | 0.20 | 0.06 | 0.06 | 0.06 | 0.06 |
| 2 | 40 | 0.1644 | 0.20 | 1.27 | 1.23 | 1.27 | 1.27 |
| 3 | 45 | 0.1644 | 0.20 | 5.35 | 5.34 | 5.18 | 5.21 |
| 4 | 50 | 0.1644 | 0.20 | 10.34 | 10.33 | 10.08 | 10.09 |
| 5 | 35 | 0.2466 | 0.20 | 0.12 | 0.12 | 0.12 | 0.13 |
| 6 | 40 | 0.2466 | 0.20 | 1.43 | 1.45 | 1.43 | 1.44 |
| 7 | 45 | 0.2466 | 0.20 | 5.30 | 5.29 | 5.09 | 5.05 |
| 8 | 50 | 0.2466 | 0.20 | 10.26 | 10.25 | 9.99 | 10.00 |
| 9 | 35 | 0.3288 | 0.20 | 0.18 | 0.19 | 0.18 | 0.19 |
| 10 | 40 | 0.3288 | 0.20 | 1.56 | 1.57 | 1.55 | 1.56 |
| 11 | 45 | 0.3288 | 0.20 | 5.31 | 5.25 | 5.04 | 5.10 |
| 12 | 50 | 0.3288 | 0.20 | 10.26 | 10.18 | 9.99 | 10.00 |
| 13 | 35 | 0.1644 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 |
| 14 | 40 | 0.1644 | 0.30 | 1.89 | 1.91 | 1.90 | 1.90 |
| 15 | 45 | 0.1644 | 0.30 | 5.48 | 5.52 | 5.45 | 5.42 |
| 16 | 50 | 0.1644 | 0.30 | 10.34 | 10.34 | 10.11 | 10.13 |
| 17 | 35 | 0.2466 | 0.30 | 0.49 | 0.49 | 0.49 | 0.50 |
| 18 | 40 | 0.2466 | 0.30 | 2.18 | 2.21 | 2.19 | 2.20 |
| 19 | 45 | 0.2466 | 0.30 | 5.58 | 5.62 | 5.55 | 5.52 |
| 20 | 50 | 0.2466 | 0.30 | 10.29 | 10.27 | 10.00 | 10.02 |
| 21 | 35 | 0.3288 | 0.30 | 0.66 | 0.67 | 0.66 | 0.67 |
| 22 | 40 | 0.3288 | 0.30 | 2.43 | 2.44 | 2.42 | 2.43 |
| 23 | 45 | 0.3288 | 0.30 | 5.71 | 5.73 | 5.64 | 5.56 |
| 24 | 50 | 0.3288 | 0.30 | 10.32 | 10.23 | 10.00 | 10.05 |

*Notes:* [a]The notation used in this table is as follows: S is the stock price, r is the riskless rate of interest, D is the amount of the cash dividend, t is the time to ex-dividend, X is the exercise price of the option, T is the time to expiration of the option, and σ is the standard deviation of the instantaneous rate of return on the stock.

[b]Blomeyer [3] dividend interpolation using Johnson [6] put approximation.

[c]Blomeyer [3] dividend interpolation using Geske–Johnson [5] put approximation.

[d]True option value computed using implicit finite difference method from Barone-Adesi and Whaley [1].

[e]Proposed nonlinear interpolation using MacMillan [7] put approximation.

true American put option, as computed using the implicit finite difference method, and the two approximations recommended by Blomeyer [3] are also presented. The parameters used to generate the option values are the same as those used by Blomeyer [3, p. 232] to facilitate comparisons. The results are quite interesting.

Table 1 contains the results for the most plausible set of parameters used in the Blomeyer analysis. A quarterly cash dividend of $.50 implies an annual dividend of $2.00, and, with a $40 share price, the annualized

*Table 2.* Comparison of American Put Option Approximation Values[a] (S = 40, r = .10, D = 1.70, t = T/2)

| *No.* | *X* | *T* | $\sigma$ | *APPX1*[b] | *APPX2*[c] | *True Value*[d] | *APPX3*[e] |
|---|---|---|---|---|---|---|---|
| 25 | 35 | 0.1644 | 0.20 | 0.13 | 0.14 | 0.13 | 0.14 |
| 26 | 40 | 0.1644 | 0.20 | 1.95 | 1.96 | 1.95 | 1.96 |
| 27 | 45 | 0.1644 | 0.20 | 6.34 | 6.33 | 6.33 | 6.35 |
| 28 | 50 | 0.1644 | 0.20 | 11.28 | 11.28 | 11.27 | 11.28 |
| 29 | 35 | 0.2466 | 0.20 | 0.22 | 0.23 | 0.23 | 0.24 |
| 30 | 40 | 0.2466 | 0.20 | 2.04 | 2.07 | 2.06 | 2.08 |
| 31 | 45 | 0.2466 | 0.20 | 6.21 | 6.19 | 6.18 | 6.24 |
| 32 | 50 | 0.2466 | 0.20 | 11.08 | 11.07 | 11.06 | 11.07 |
| 33 | 35 | 0.3288 | 0.20 | 0.30 | 0.32 | 0.32 | 0.33 |
| 34 | 40 | 0.3288 | 0.20 | 2.12 | 2.15 | 2.15 | 2.17 |
| 35 | 45 | 0.3288 | 0.20 | 6.10 | 6.07 | 6.07 | 6.16 |
| 36 | 50 | 0.3288 | 0.20 | 10.88 | 10.86 | 10.86 | 10.88 |
| 37 | 35 | 0.1644 | 0.30 | 0.47 | 0.47 | 0.47 | 0.48 |
| 38 | 40 | 0.1644 | 0.30 | 2.49 | 2.54 | 2.53 | 2.53 |
| 39 | 45 | 0.1644 | 0.30 | 6.46 | 6.49 | 6.49 | 6.48 |
| 40 | 50 | 0.1644 | 0.30 | 11.30 | 11.29 | 11.28 | 11.30 |
| 41 | 35 | 0.2466 | 0.30 | 0.69 | 0.70 | 0.70 | 0.71 |
| 42 | 40 | 0.2466 | 0.30 | 2.74 | 2.79 | 2.79 | 2.79 |
| 43 | 45 | 0.2466 | 0.30 | 6.44 | 6.51 | 6.51 | 6.49 |
| 44 | 50 | 0.2466 | 0.30 | 11.14 | 11.13 | 11.12 | 11.16 |
| 45 | 35 | 0.3288 | 0.30 | 0.88 | 0.89 | 0.89 | 0.90 |
| 46 | 40 | 0.3288 | 0.30 | 2.93 | 3.00 | 3.00 | 3.01 |
| 47 | 45 | 0.3288 | 0.30 | 6.46 | 6.55 | 6.54 | 6.54 |
| 48 | 50 | 0.3288 | 0.30 | 11.09 | 11.00 | 10.99 | 11.06 |

*Notes:* [a]The notation used in this table is as follows: S is the stock price, r is the riskless rate of interest, D is the amount of the cash dividend, t is the time to ex-dividend, X is the exercise price of the option, T is the time to expiration of the option, and $\sigma$ is the standard deviation of the instantaneous rate of return on the stock.

[b]Blomeyer [3] dividend interpolation using Johnson [6] put approximation.

[c]Blomeyer [3] dividend interpolation using Geske–Johnson [5] put approximation.

[d]True option value computed using implicit finite difference method from Barone-Adesi and Whaley [1].

[e]Proposed nonlinear interpolation using MacMillan [7] put approximation.

dividend yield is 5%. With a $.50 dividend paid half way through the option's life, the proposed nonlinear interpolation values (7) or (8) reported in the column headed "APPX3" are remarkably close to the true values reported in the column headed "True Value." The largest errors tend to occur where the put is slightly in-the-money, but even in these cases the pricing errors are only about 1% and are less, in magnitude, than the bid–ask spread for such options.[10]

Another way in which the performance of the nonlinear interpolation can be gauged is by comparing its accuracy with the approximation techniques recommended by Blomeyer. Blomeyer's values are reported as columns "APPX1" and "APPX2" in Table 1. Without going into the details of the Blomeyer computations, it is obvious that his values are very misleading, with errors on order of 2–3%.[11]

Blomeyer [3, p. 232] also reports the results of his option price approxi-

*Table 3.* Comparison of American Put Option Approximation Value[a] (S = 40, r = .10, D = 0.40, T = .25)

| | | | $\sigma = 0.20$ | | $\sigma = 0.30$ | | $\sigma = 0.40$ | |
|---|---|---|---|---|---|---|---|---|
| *No.* | *X* | *t* | *True Value*[b] | *APPX3*[c] | *True Value*[b] | *APPX3*[c] | *True Value*[b] | *APPX3*[c] |
| 1 | 35 | 0.05 | 0.12 | 0.12 | 0.48 | 0.49 | 1.00 | 1.01 |
| 2 | 40 | 0.05 | 1.41 | 1.40 | 2.17 | 2.17 | 2.94 | 2.94 |
| 3 | 45 | 0.05 | 5.22 | 5.24 | 5.58 | 5.53 | 6.17 | 6.14 |
| 4 | 50 | 0.05 | 10.15 | 10.15 | 10.18 | 10.19 | 10.40 | 10.33 |
| 5 | 35 | 0.10 | 0.12 | 0.12 | 0.48 | 0.49 | 1.00 | 1.01 |
| 6 | 40 | 0.10 | 1.41 | 1.40 | 2.16 | 2.17 | 2.94 | 2.94 |
| 7 | 45 | 0.10 | 5.07 | 5.04 | 5.52 | 5.47 | 6.14 | 6.12 |
| 8 | 50 | 0.10 | 10.00 | 10.00 | 10.00 | 10.02 | 10.29 | 10.18 |
| 9 | 35 | 0.15 | 0.12 | 0.12 | 0.48 | 0.49 | 1.00 | 1.00 |
| 10 | 40 | 0.15 | 1.37 | 1.38 | 2.15 | 2.15 | 2.92 | 2.93 |
| 11 | 45 | 0.15 | 5.00 | 5.06 | 5.43 | 5.36 | 6.08 | 5.99 |
| 12 | 50 | 0.15 | 9.99 | 10.00 | 9.99 | 10.02 | 10.19 | 10.13 |
| 13 | 35 | 0.20 | 0.11 | 0.12 | 0.48 | 0.49 | 0.99 | 1.00 |
| 14 | 40 | 0.20 | 1.34 | 1.35 | 2.12 | 2.11 | 2.90 | 2.88 |
| 15 | 45 | 0.20 | 4.99 | 5.04 | 5.37 | 5.30 | 6.02 | 5.90 |
| 16 | 50 | 0.20 | 9.99 | 10.00 | 9.99 | 10.00 | 10.14 | 10.10 |

*Notes:* [a]The notation used in this table is as follows: S is the stock price, r is the riskless rate of interest, D is the amount of the cash dividend, t is the time to ex-dividend, X is the exercise price of the option, T is the time to expiration of the option, and $\sigma$ is the standard deviation of the instantaneous rate of return on the stock.

[b]True option value computed using implicit finite difference method from Barone-Adesi and Whaley [1].

[c]Proposed nonlinear interpolation using MacMillan [7] put approximation.

mations for the case where the quarterly dividend is 1.70, and, for convenience, they are reported in Table 2. Here the Blomeyer approximations work about equally as well as the nonlinear interpolation, however, these results are of little practical value. A quarterly dividend of $1.70 on a $40 share price implies an annual dividend yield of a whopping 17%! This is not a typical common stock by any means.

Table 3 contains the simulation results for a different set of parameters. The parameters were chosen to correspond with more typical stock options. The dividend amount is $.40 on a $40 share price, implying an annual dividend yield of 4%. The standard deviation of the rate of return on the stock varies from 20 to 40%, and the times to ex-dividend vary from 0.05 to 0.20 years. The time to expiration is 0.25 years because the most actively traded stock options are typically the nearby contracts and the maximum time to expiration of a nearby contract is 3 months.

The results of Table 3 indicate that the nonlinear interpolation does reasonably well for the parameter ranges considered. Again, the put options slightly in-the-money have the greatest mispricing errors, but the magnitudes of the errors again fall within transaction costs bands. Overall, the nonlinear approximation works remarkably well considering that it takes less than 1/1000th of the time to compute the nonlinear interpolation than it does the finite difference value of these options.[12]

## IV. SUMMARY

Pricing American put options on dividend-paying stocks has largely been ignored in the literature because the problem is mathematically complex and valuation usually resorts to expensive approximation procedures. This paper provides a simple and fast nonlinear interpolation procedure that yields surprisingly accurate results.

## NOTES

1. Stoll and Whaley [11] develop a general framework for valuing options whose underlying commodities have constant continuous cost of carry rates. Barone-Adesi and Whaley [1] provide algorithms for pricing American options on such commodities.

2. Equation (2) is only an approximation of the true value of the American put option P(S, T). It was chosen over competing approximation methods (see, for example, Johnson [6] and Geske–Johnson [5]) because of its speed and accuracy. Johnson's method is fast but generally unreliable (see Barone-Adesi and Whaley [1] for evidence on this point). The Geske–Johnson compound option approach is accurate when a four-point extrapolation is used, but it requires the evaluation of a trivariate normal density function integral and routines for this integral evaluation are generally slow.

3. To be more precise, the early exercise premium increases first at an increasing, then at a decreasing, rate as the time to expiration of the option grows large. The range of times to expiration over which the slope is increasing, however, is very close to zero and not relevant to the options being considered here. In general, the rate at which the premium increases diminishes as the time to expiration grows large because the American put option value approaches its asympotic limit. See Merton [8, pp. 173–174].

4. For computational purposes, Equation (7) may be simplified to

$$P(S,T;D,t_D) = P(S',T) - w_2\varepsilon_P(S',t_D).$$

5. The probabilities computed here are risk-neutral probabilities in the spirit of Cox and Ross [4].

6. If the put is deep in-the-money, the American put option values, $P(S',T)$ in (6) and $P(S',t_D)$ in (6a), equal $X-S$. Thus, the value of (6) is $X-S-[X-S-p(S',t_D)]$ or simply $p(S',t_D)$.

7. For computational purposes, Equation (8) may be simplified to

$$P(S,T;D,t_D) = (w_1+w_2)P(S',T) - w_2\varepsilon_P(S',t_D) + w_3P(S,t_N).$$

8. This critical stock price is computed implicitly for the American put option value $P(S,T-t_N)$ using the valuation equation (2).

9. An algorithm for evaluating the bivariate normal integral is contained in Owen [9]. Alternatively, since the ratio $t_N/t_D$ is usually small, reasonable approximations of the bivariate probabilities $N_2(a,b;\sqrt{t_N/t_D})$ and $N_2(a,-b;-\sqrt{t_N/t_D})$ are the products of the univariate probabilities $N_1(a)N_1(b)$ and $N_1(a)\,N_1(-b)$, respectively.

10. The minimum bid–ask spread for options whose prices exceed \$3 is one-eighth or \$0.125. The minimum spread for options whose prices are below \$3 is one-sixteenth or \$0.0625.

11. Generally speaking, as the dividend becomes small or as the time to ex-dividend grows large, the Blomeyer approximations become worse.

12. The finite difference method used time steps of 0.20 days and stock prices steps of \$0.04.

# REFERENCES

1. Barone-Adesi, G., and R. E. Whaley, "Efficient Analytic Approximation of American Option Values," *Journal of Finance* 42 (June 1987), 301–320.
2. Black, F., and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (May–June 1973), 637–659.
3. Blomeyer, E. C., "An Analytical Approximation for the American Put Price for Options on Stocks with Dividends," *Journal of Financial and Quantitative Analysis* 21 (June 1986), 229–233.
4. Cox, J. C., and S. A. Ross, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics* 3 (1986), 145–166.
5. Geske, R., and H. E. Johnson, "The American Put Valued Analytically," *Journal of Finance* 39 (December 1984), 1511–1524.
6. Johnson, J., "An Analytical Approximation for the American Put Price," *Journal of Financial and Quantitative Analysis* 18 (March 1983), 141–148.
7. MacMillan, L. W., "Analytical Approximation for the American Put Option," *Advances in Futures and Options Research* 1 (1986), 119–139.

8. Merton, R. C., "The Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4 (1973), 141–183.
9. Owen, D. B., *Handbook of Statistical Tables.* Reading, Massachusetts: Addison-Wesley Publishing Company, 1962.
10. Roll, R., "An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends," *Journal of Financial Economics* 5 (1977), 251–258.
11. Stoll, H. R., and R. E. Whaley, "The New Option instruments: Arbitrageable Linkages and Valuation," *Advances in Futures and Option Research* 1 (1986), 25–62.
12. Whaley, R. E., "On the Valuation of American Call Options on Stocks with Known Dividends," *Journal of Financial Economics* 9 (1981), 207–211.