Journal of Financial Economics 9 (1981) 207–211. North-Holland Publishing Company

ON THE VALUATION OF AMERICAN CALL OPTIONS ON STOCKS WITH KNOWN DIVIDENDS

Robert E. WHALEY*

Vanderbilt University, Nashville, TN 37203, USA

Received August 1980, final version received October 1980

Both the Roll and the Geske equations for the valuation of the American call option on a stock with known dividends are incorrectly specified. This note presents the corrected valuation formula, explains the misspecifications and provides a numerical example.

In two recent articles of this *Journal*, Roll (1977) and Geske (1979b) present valuation formulae for an unprotected American call option on a stock with known dividends. While both authors demonstrate a great deal of ingenuity in their approaches to finding the solution to what was deemed to be an unmanageable pricing problem, their valuation equations are not correctly specified. This note presents the corrected valuation formula, compares it to the Roll and Geske models, and explains how their misspecification errors arise. To assist those who may attempt to implement this option pricing relationship, a numerical example is provided.

The assumptions underlying the valuation of the American call are:

(a) All individuals can borrow or lend without restriction at the instantaneous riskless rate of interest, r, and that rate is constant through the life of the option, T.

(b) At the ex-dividend instant, t $(t<T)$, the stock pays a dividend of D which induces a stock price decline of αD.

(c) The stock price net of the escrowed dividend, S $(S_\tau = P_\tau - \alpha D\,\mathrm{e}^{-r(t-\tau)}$ for $\tau < t$ and $S_\tau = P_\tau$ for $t \leqq \tau \leqq T$, where P is the stock price *cum* dividend), is described by the stochastic differential equation

$$\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}z,$$

where μ is the instantaneous expected rate of return on the common stock, σ is the instantaneous standard deviation of stock price return

*Comments by Richard Roll are gratefully acknowledged.

(assumed to be constant over the life of the option), and dz is a standard unit normally distributed variable.

Additionally, an assumption of perfect capital markets is invoked.

Under the assumptions listed above there exists some finite ex-dividend stock price S_t^* above which the option will be exercised early.[1] It is found by applying a numerical search procedure to

$$c(S_t^*, T-t, X) = S_t^* + \alpha D - X,$$

where X is the exercise price of the option and $c(\cdot)$ is the market value of a European call, as provided by Black and Scholes (1973). If, just prior to the ex-dividend instant, $S_t > S_t^*$, the American option holder will exercise realizing cash proceeds of $S_t + \alpha D - X$. If, on the other hand, $S_t \leqq S_t^*$, the owner will choose to hold his position open until expiration since the option is worth more unexercised.

Once the value of S_t^* is established, the Roll (1977) and Ross (1978) valuation by duplication technique can be applied to solve the option pricing problem. For example, consider the following portfolio of options:

(a) a long position of one European call option with exercise price X and maturity T;
(b) a long position of one European call option with exercise price S_t^* and maturity $t-\varepsilon$ ($\varepsilon > 0, \varepsilon \simeq 0$); and
(c) a short position in one European call option on the option described by (a) with exercise price $S_t^* + \alpha D - X$ and maturity $t-\varepsilon$.

Since the income contingencies of this portfolio are identical to those posed by the American call option, the absence of costless arbitrage opportunities in a perfect capital market ensures that the market value of the American call is equal to the market value of the portfolio. Applying the Black–Scholes (1973) option pricing formula to options (a) and (b) of the portfolio and the Geske (1979a) compound option pricing formula to (c), substituting the identity $N_2(a, -b; -\rho) = N_1(a) - N_2(a, b; \rho)$,[2] and gathering terms on S, X and αD, the value of an American call option on a stock with a *single* known dividend paid during the life of the option is

$$\begin{aligned} C(S,T,X) = {} & S[N_1(b_1) + N_2(a_1, -b_1; -\sqrt{t/T})] \\ & - X\,e^{-rT}[N_1(b_2)\,e^{r(T-t)} + N_2(a_2, -b_2; -\sqrt{t/T})] \\ & + \alpha D\,e^{-rt} N_1(b_2), \end{aligned} \tag{1}$$

[1] S_t^* is infinite, and early exercise is not possible, when $\alpha D \leqq X(1-e^{-rt})$. See Roll (1977, p. 252).

[2] For an excellent discussion about the nature of the bivariate normal integral, see Johnson and Kotz (1972, pp. 93–100).

where

$$a_1 = \frac{\ln(S/X) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \qquad a_2 = a_1 - \sigma\sqrt{T},$$

$$b_1 = \frac{\ln(S/S_t^*) + (r + 0.5\sigma^2)t}{\sigma\sqrt{t}}, \qquad b_2 = b_1 - \sigma\sqrt{t},$$

$N_1(a)$ is the univariate cumulative normal density function with upper integral limit a and $N_2(a, b; \rho)$ is the bivariate cumulative normal density function with upper integral limits a and b and correlation coefficient ρ.

It should be noted that there is nothing unique about the set of options used to price the American call. Other portfolios, some, in fact, with fewer options, will work equally as well. Roll's (1977, p. 254) portfolio, however, will not. He expresses the value of the American call as the sum of the values of the following three options:

(a) c_a, a European call option with exercise price X and maturity T;
(b) c_b, a European call option with exercise price $S_t^* + \alpha D$ and maturity $t - \varepsilon$; minus
(c) c_c, a European call option on the option described under (a) with exercise price $S_t^* + \alpha D - X$ and maturity $t - \varepsilon$.

Algebraically,

$$C_{RLL} = c_a + c_b - c_c, \tag{2}$$

where

$$c_a = SN_1(a_1) - X\,e^{-rT}N_1(a_2), \tag{2a}$$

$$c_b = SN_1(d_1) - (S_t^* + \alpha D)\,e^{-rt}N_1(d_2), \tag{2b}$$

$$c_c = SN_2(a_1, b_1; \sqrt{t/T}) - X\,e^{-rT}N_2(a_2, b_2; \sqrt{t/T}) - (S_t^* + \alpha D - X)\,e^{-rt}N_1(b_2), \tag{2c}$$

$$d_1 = \frac{\ln[S/(S_t^* + \alpha D)] + (r + 0.5\sigma^2)t}{\sigma\sqrt{t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{t}.$$

The misspecification in the value of C_{RLL} is contained within the second option (b): its exercise price should be S_t^* instead of $S_t^* + \alpha D$ unless it is P, the stock price *cum* dividend, that follows the log-normal diffusion process.

This option should have cash proceeds of $S_t+\alpha D-S_t^*-\alpha D$ at the ex-dividend instant if the option is exercised just prior to ex-dividend. An option whose value is described by $c(P,t,S_t^*+\alpha D)$ has such proceeds. At t, the option holder receives

$$P_t-S_t^*-\alpha D \quad \text{if} \quad P_t>S_t^*+\alpha D,$$

and

$$0 \quad \text{if} \quad P_t \leqq S_t^*+\alpha D.$$

Since $S_t \simeq P_t-\alpha D$ at $t-\varepsilon$, these conditions can be alternatively expressed as

$$S_t-S_t^* \quad \text{if} \quad S_t>S_t^*,$$

and

$$0 \quad \text{if} \quad S_t \leqq S_t^*.$$

The appropriate valuation equation for this latter set of conditions is

$$c_b=SN_1(b_1)-S_t^*\,\mathrm{e}^{-rt}N_1(b_2). \tag{2b'}$$

Replacing c_b as it described by eq. (2b) with c_b from (2b′), substituting the aforementioned identity, and gathering terms on S, X and αD, the corrected Roll model simplifies to eq. (1).

By solving the option pricing problem directly, Geske (1979) values the American call as

$$\begin{aligned} C_{GES}=&\,S[N_1(b_1)+N_2(a_1,c_1;\sqrt{t/T})] \\ &-X\,\mathrm{e}^{-rT}[N_1(b_2)\mathrm{e}^{r(T-t)}+N_2(a_2,c_2;\sqrt{t/T})] \\ &+\alpha D\,\mathrm{e}^{-rt}N_1(b_2), \end{aligned} \tag{3}$$

where

$$c_1=\frac{\ln(S_t^*/S)-(r+0.5\sigma^2)t}{\sigma\sqrt{t}} \quad \text{and} \quad c_2=c_1+\sigma\sqrt{t}.$$

The problem with this specification is that the correlation coefficient should read $-\sqrt{t/T}$ rather than $\sqrt{t/T}$. When the correct value is substituted, eq. (3) is equivalent to (1) since the values of c_1 and c_2 are simply $-b_1$ and $-b_2$, respectively.

To assist those implementing this option pricing model, consider the following illustration. All computations are performed on an American call option with the parameters $X=100$, $t=1$, $T=2$, $r=0.04$, $\sigma=0.20$ and $\alpha D=5$.

The Black (1975, p. 41) dividend approximation technique values, $c(S,T,X)$, are included to facilitate comparison. Note that the values of $C(S,T,X)$ are not (and, in fact, cannot be) below the Black approximation values. Direct application of the pricing equations (2) and (3), however, does not ensure this result.

Table 1

Hypothetical American call option values.[a]

Stock price *cum* dividend P	Stock price *ex* dividend S	Indifference price S_t^*	American call option value $C(S,T,X)$	Black approximation $c(S,T,X)$
80.0	75.196	123.582	3.212	3.208
85.0	80.196	123.582	4.818	4.808
90.0	85.196	123.582	6.839	6.820
95.0	90.196	123.582	9.276	9.239
100.0	95.196	123.582	12.111	12.048
105.0	100.196	123.582	15.316	15.215
110.0	105.196	123.582	18.851	18.703
115.0	110.196	123.582	22.676	22.470
120.0	115.196	123.582	26.748	26.476

[a]The American call option contract in the above illustration is assumed to have an exercise price of 100 and a time to expiration of 2 periods. The underlying common stock has a rate of return standard deviation of 0.2 and a known stock price decline of 5 at the end of 1 period. The riskless rate of return is 4 percent.

In summary, both the Roll and the Geske equations for the valuation of the American call option on a stock with known dividends are incorrectly specified. This note presents the corrected valuation formula, explains the misspecifications and provides a numerical example.

References

Black, F., 1975, Fact and fantasy in the use of options, Financial Analysts Journal 31, 36–41, 61–72.

Black, F. and M. Scholes, 1973, The pricing of options and corporate liabilities, Journal of Political Economy 81, 637–659.

Geske, R., 1979a, The valuation of compound options, Journal of Financial Economics 7, 63–81.

Geske, R., 1979b, A note on an analytical valuation formula for unprotected American call options on stocks with known dividends, Journal of Financial Economics 7, 375–380.

Johnson, N.L. and S. Kotz, 1972, Distributions in statistics: Continuous multivariate distributions (Wiley, New York).

Roll, R., 1977, An analytical valuation formula for unprotected American call options on stocks with known dividends, Journal of Financial Economics 5, 251–258.

Ross, S., 1978, A simple approach to the valuation of risky streams, Journal of Business 51, 453–475.